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Vortex ring eigen-oscillations as a source of sound

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Two coupled problems are investigated: a complete description of long-wave vortex ring oscillations in an ideal incompressible fluid, and an examination of sound radiation by these oscillations in a weakly compressible fluid.

The first part of the paper relates to the problem of eigen-oscillations of a thin vortex ring ($\mu \ll 1$) in an ideal incompressible fluid. The solution of the problem is obtained in the form of an asymptotic expansion in the small parameter μ. The complete set of three-dimensional eigen-oscillations and axisymmetric modes (two-dimensional oscillations) is obtained. It is shown that, unlike the vortex column oscillations which have the form of simple angular harmonics, the majority of eigen-oscillations of a thin vortex ring have a more complex form which is a combination of two harmonics in the leading approximation. This leads to dramatic changes in the efficiency of sound radiation produced by modes of the vortex ring in comparison with the corresponding modes of the vortex column.

In the second part of the paper the solution obtained is used to investigate the process of sound radiation by vortex perturbations in a weakly compressible fluid. The vortex ring eigen-oscillations are classified according to their sound radiation efficiency. It is shown that the modes with the dimensionless frequency $\omega \approx 1/2$ radiate sound most efficiently. They are two isolated modes, two infinite families of Bessel modes and a set of axisymmetric modes. The frequencies of these modes are in the interval $\Delta\omega = O(\mu)$.

The results obtained are compared with known experimental data on acoustic radiation of a turbulent vortex ring. Within the limits of the theory derived an explanation of the main characteristics of sound radiation is presented.

1. Introduction

The problem of vortex ring oscillations in an ideal incompressible fluid has attracted continuous interest starting from the first solutions obtained in the last century (Kelvin 1867; Thomson 1883) as applied to the vortex theory of an atom. Owing to the extreme complexity of the problem all theoretical works on this subject have been restricted to the case of a thin vortex ring ($\mu \ll 1$, μ is the ratio of the vortex cross-section to the ring radius). However, even for this case only those eigen-oscillations are obtained which have a structure that allows the task to be simplified still more.

The main approaches developed in this problem will now be summarized.

The situation seems to be the most simple for axisymmetric oscillations, which reduces the problem to an evaluation of the disturbed vortex boundary only. This reduction permits both small (Basset 1888; Kopiev & Leontiev 1987) and nonlinear disturbances (Moore 1980) to be investigated.

For short-wave oscillations in the azimuthal angle θ (Widnall & Tsai 1977) the wavelength is an additional small parameter which permits the neglect of the mutual influence of disturbances in ring regions distant from each other. A careful analysis of the physical mechanism of short-wave instability is given by Saffman (1992).

The presence of the small parameter μ underlies two well-known approaches in which the vortex ring oscillation problem is replaced by the problem corresponding to the limit $\mu = 0$. This limit can be realized in two different ways: first it corresponds to finite radius of the ring and zero size of the core cross-section, that is the flow structure inside the vortex core is neglected; secondly it corresponds to infinite radius of the ring and finite size of the core cross-section, that is the mean curvature of the undisturbed ring is neglected.

The first case corresponds to the localized induction approximation (LIA) (Hama 1963) or a more precise one (Crow 1970; Widnall, Bliss & Zalay 1971; Klein & Majda 1991). In both these approximations the self-induced velocity at some section is calculated as if the whole vortex were a filament of infinitesimal thickness except the near section considered where the flow structure is assumed to be coincident with the flow structure of the osculatory steady vortex ring. In the problem of vortex ring oscillations these solutions are reported to describe the evolution of the mean line only (bending mode). However, as is shown in the present work, the mean line evolution can be determined not only by its own form but by the structure of the disturbances inside vortex core, since a particular disturbance of the mean line may correspond to quite different modes of the vortex ring.

In the second case the limit $\mu = 0$ is achieved for a constant core cross-section (Basset 1888; Ladikov 1960). This limit corresponds in reality to replacing the vortex ring by a vortex column. A comprehensive analysis carried out in the present paper shows that the vortex ring oscillations have a frequency spectrum which is similar to a vortex column spectrum. However, the vortex column approximation gives an incorrect form for the eigen-oscillations.

In the first part of the paper a complete description of long-wave oscillations of a thin ($\mu \ll 1$) vortex ring is presented (§§3 and 4). For a careful statement of the problem of vortex ring oscillations the choice of the stationary flow giving the most simple solution of this cumbersome problem is important. For thin vortex rings there is an infinite set of vorticity distributions in the core for which the flow is steady in the system of coordinates attached to the vortex ring (Fraenkel 1970). The most simple of them is the uniform vorticity distribution for which the vorticity vector amplitude is proportional to the distance from the vortex symmetry axis ($\Omega/\xi = \text{const}$). The steady flow with such a vorticity distribution exists not only for thin ($\mu \ll 1$) rings but also for rings with arbitrary values of μ (Norbury 1973) including Hill's vortex (Milne-Thompson 1960). Note that for a uniform vorticity distribution the period of fluid particle orbits is not a constant over the whole vortex core cross-section (non-isochronism). Non-isochronism of fluid particle orbits can be readily seen in an example of the limiting case – Hill's vortex, for which the period of rotation tends to infinity on approaching the vortex boundary. The non-isochronism of fluid particles leads to the appearance of a critical layer and the emergence of continuous spectrum disturbances. Such a situation does not appear in the case of a vortex column where the conditions of isochronism and of vorticity uniformity are identical. However, for a thin vortex ring the vorticity distribution fulfilling the condition of isochronism is close to the uniform one, differing from it only by terms of $O(\mu^2)$ (§4.1).

The vortex ring with uniform vorticity is used in this paper to examine axisymmetric disturbances. This steady flow is most convenient because in this case a vortex boundary displacement does not produce vorticity disturbances inside the vortex core.

Axisymmetric oscillations are considered in §3 where, beside the well-known discrete mode, the modes of the continuous spectrum are also considered. However, for three-dimensional oscillations consideration of continuous spectrum disturbances is too cumbersome a task. Thus the isochronous vortex ring which has only a discrete spectrum is used for three-dimensional oscillations because this vortex ring is the simplest one from this standpoint.

To illustrate the main characteristics of vortex ring oscillations, in §2 the well-known problem of cylindrical vortex oscillations is briefly considered.

A new procedure for three-dimensional oscillation prediction is devised. The problem of obtaining eigen-oscillations is divided into three more-simple ones (§4.4). First, a set of basic disturbances is constructed in which eigen-oscillations are expanded. Secondly, the Biot–Savart integral is calculated for each basic disturbance. Thirdly, the system of algebraic equations is solved which determines forms of the eigen-oscillations and eigen-frequencies. With such an approach it is possible to obtain for each approximation both the form of boundary disturbances and the disturbance structure inside vortex and to estimate the value of the terms rejected.

To identify three-dimensional eigen-oscillations of the vortex ring we use three integers: l, n, j where the first characterizes frequency and is called a frequency number, the second characterizes the number of wavelengths fitting into the ring mean line and is called the azimuthal number, and the third, called a radial number, characterizes the oscillation structure in the core section for Bessel modes ($j \geqslant 1$), and is assumed to be equal to zero for isolated modes. The form of the isolated modes of the vortex ring (mode $(l, n, 0)$) to the leading approximation has the form of the harmonic $\exp[\mathrm{i}(l+1)\psi]$ in the vortex core section, i.e. it appears to be close to the corresponding oscillations of a cylindrical vortex. Bessel oscillations of the vortex ring, in contrast to the vortex column, to the leading approximation appear to be a linear combination of two ψ-harmonics with adjacent numbers l and $l+1$. This leads to dramatic changes in the efficiency of sound radiation produced by these modes of the vortex ring in comparison with the corresponding modes of the vortex column.

The solutions obtained (axisymmetrical and three-dimensional modes) are a complete typical set of disturbances. For long time ($T < 1/\mu^2$) these disturbances will describe the evolution of the initial disturbances of any vortex ring with steady vorticity differing from uniform or isochronous values by terms of $O(\mu^2)$ (§4.7).

The solutions obtained in the first part of the paper (§§3 and 4) are used in the second part (§5) to describe the sound fields generated by vortex rings and to classify eigen-oscillations according to their sound radiation efficiency. As is known, the unsteady movement of vortices in a compressible fluid is accompanied by a quadrupole sound radiation (Lighthill 1952). If the characteristic flow Mach number is small and vorticity is localized in some region with a characteristic size much less than the sound wavelength, then the sound field will be expressed through the unsteady velocity field which can be calculated with the approximation of an incompressible fluid. For calculation of the sound field generated by vortices it is convenient to connect the sound field with only that flow part where the vorticity differs from zero (Powell 1964). The most convenient expression for the sound field which links the unsteady vorticity field and the sound generated in a linear manner was obtained by Möhring (1978). An alternative derivation of Möhring's expression is discussed in Obermeier (1979), Kambe (1986), Kopiev & Leontiev (1987), and Powell (1995). Möhring (1978) calculated the sound field generated by two leap-frogging vortex rings. It was the first model example of a localized three-dimensional aerodynamic sound source in unbounded flow, which was based on the three-dimensional dynamics of the vorticity.

A solitary vortex ring in unbounded flow can also be a source of sound. The first expression for sound radiation by a solitary vortex ring oscillations was obtained by Kopiev & Leontiev (1987) for the most simple case of its axisymmetric oscillations. Expressions for sound radiation by bending and bulging modes are presented by Kopiev (1992). Kopiev & Chernyshev (1993) briefly consider the isolated modes for a uniform vortex ring (including an axisymmetric one) and the sound generated from them. However, they did not pay attention to the presence of the continuous spectrum in this case. Two families of Bessel modes with $n = 1$ and $n = 2$, which are the most interesting from the point of view of sound radiation, are not considered anywhere.

The main problem arising in calculation of the sound field generated by vortex ring oscillations consists in the following. The vortex ring eigen-oscillation are presented as a sum of the harmonics $\exp(\mathrm{i}m\psi)$. It turns out that the same contribution into the sound field can be made by various ψ-harmonics which have different orders of magnitude in μ. Therefore a careful analysis of the contribution of each term of the eigen-oscillations to the sound field is required to provide an assurance that all the necessary terms are taken into account.

To this end, Möhring's expression for the quadrupole moment is transformed in this paper. In this expression an integral over the vortex volume is transformed into the integral over the vortex surface (§5). Such a representation permits the evaluation in a relatively simple manner of the contribution of various harmonics to the sound field. It is shown that for simultaneous excitation of all modes of the vortex ring only the oscillations with $l = 1$ (the frequency near the value $\omega = 1/2$) will be found in the far acoustic field. They are axisymmetric modes, two isolated modes $(1, 1, 0)$, $(1, 2, 0)$ and an infinite number of Bessel modes of the type $(1, 1, j)$ and $(1, 2, j)$, $j \geqslant 1$.

Finally, (§6) the results obtained are compared with the known experimental data on acoustic radiation of a turbulent vortex ring. Within the limits of the theory derived an explanation of the main characteristics of sound radiation is presented.

2. Oscillations of a vortex column

In the limiting case of $R \to \infty$, $a = \text{const}$ (R is the vortex ring radius, a is the radius of the vortex core cross-section) the vortex ring flow corresponds to that of a cylindrical vortex. Small oscillations of a cylindrical vortex with a constant vorticity Ω_0 and without axial flow (vortex column) have been obtained by Kelvin (1880) and are well-known (Saffman 1992). Nevertheless, a short review of these oscillations is given in this Section, since the forms and frequencies of the vortex column oscillations are a starting point in the examination of the vortex ring oscillations.

The eigen-oscillations of the cylindrical vortex have the form

$$v^i = v^i(\rho)\,\mathrm{e}^{\mathrm{i}m\phi + \mathrm{i}ks - \mathrm{i}\omega t}, \quad m = 0, 1, 2, \ldots, \tag{2.1}$$

where ρ, ϕ, s are the cylindrical coordinates with the axis $\boldsymbol{e}_s$ along the vortex axis. The possibility of looking for the solution in this form is associated with the cylindrical vortex symmetry relative to translations along s and rotations around $\boldsymbol{e}_s$.

Only cylindrical vortex oscillations with $k \geqslant 0$ will be considered here. The solutions with $k < 0$ can be easily obtained from the solutions with $k > 0$ through substitution of $v^s \to -v^s$ and $k \to -k$ in the amplitudes $v^i(\rho)$. In the following equations dimensionless variables are used. The time scale is Ω_0^{-1}. The length scale is a.

The dispersion equation for cylindrical vortex oscillations has the form

$$\frac{\omega' k J_{m+1}(k(1-\omega'^2)^{1/2}/\omega')}{(1-\omega'^2)^{1/2} J_m(k(1-\omega'^2)^{1/2}/\omega')} + \frac{\mathrm{i}k H^{(1)}_{m+1}(\mathrm{i}k)}{H^{(1)}_m(\mathrm{i}k)} - \frac{m}{1+\omega'} = 0, \tag{2.2}$$

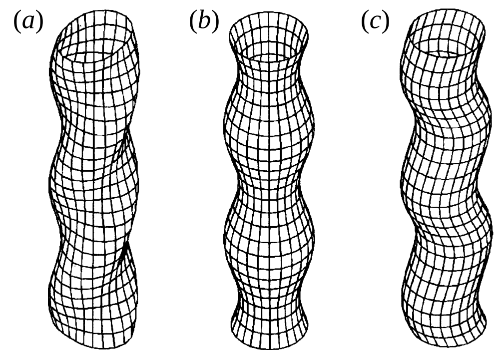

FIGURE 1. Boundary disturbance shapes of a cylindrical vortex. (*a*) Bessel modes and an isolated mode at $m = 2$; (*b*) bulging mode; (*c*) bending mode.

where $\omega' = \omega - m/2$.

This equation is valid for arbitrary k; however, below we shall consider only the long-wave case $k \ll 1$. At fixed m and $k \ll 1$ (2.2) has solutions in two ranges: if $k(1-\omega'^2)^{1/2}/\omega' = O(1)$ and if $k(1-\omega'^2)^{1/2}/\omega' \ll 1$, i.e. the long-wave oscillations ($k \ll 1$) of the cylindrical vortex are concentrated in two frequency ranges: $\omega' \ll 1$ and $\omega' = O(1)$. In the first case $\omega' \ll 1$

$$\omega_j = \frac{m}{2} \pm \frac{k}{a_j} + O(k^2), \quad j = 1, 2, \ldots, \quad m = 0, 1, 2, \ldots, \tag{2.3}$$

where the a_j are the zeros of the Bessel function $J_m(a)$. The corresponding eigen-oscillations will be referred to as *Bessel* modes. The eigen-frequencies ω_j of these modes have the point of accumulation $\omega = m/2$. In the second case $\omega' = O(1)$ (2.2) has one more solution for each $m \neq 0$:

$$\omega = -\frac{k^2}{4}\left(\ln\frac{2}{k} - C + \frac{1}{4}\right) + O(k^4 \ln k), \quad m = 1, \tag{2.4}$$

$$\omega = \frac{m-1}{2} - \frac{k^2}{4(m-1)(m+1)} + O(k^3), \quad m \geqslant 2, \tag{2.5}$$

where $C \approx 0.58$ is Euler's constant (Abramowitz & Stegun 1965). The corresponding eigen-oscillation will be referred to as an *isolated* mode.

It follows from (2.1) that at any moment of time the disturbed vortex boundary has an m-fold form at each section $s = \text{const}$. This form rotates around the cylinder axis at an angle depending on s according to the expression $\exp(\mathrm{i}m\phi + \mathrm{i}ks)$ (figure 1).

For Bessel modes (2.3) the velocity disturbances inside the vortex are determined by Bessel functions and have an oscillating form in the coordinate ρ. The number of these oscillations in ρ increases as j increases. Bessel modes with $m = 0$ are characterized by periodical variations of the vortex cross-section area along the axis $\boldsymbol{e}_s$. The vortex boundary in this case has a specific bulging form (figure 1*b*). In this connection, the Bessel mode with $m = 0$ is referred to as a *bulging* mode.

The disturbance form for an isolated mode (2.4), (2.5) presented in figure 1(*a*, *c*) is of power type in the coordinate ρ inside the vortex. The isolated mode with $m = 1$ is called the *bending* mode since in the leading approximation this oscillation reduces to a periodical displacement of the vortex mean line (figure 1*c*).

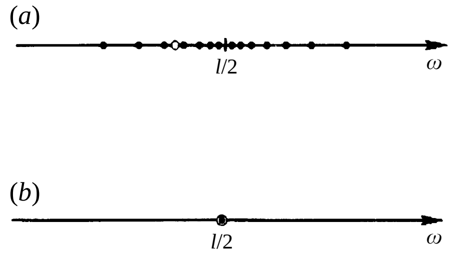

FIGURE 2. Arrangement of the eigen-frequencies of the cylindrical vortex oscillations near $\omega = l/2$; (*a*) $k > 0$; (*b*) $k = 0$. ●, Bessel modes at $m = l$; ○, an isolated mode at $m = l+1$.

The important distinction between Bessel and isolated modes concerns the character of the velocity disturbance outside the vortex. For the isolated oscillations the disturbance amplitude outside the vortex has the same order of magnitude as the displacement amplitude of the vortex boundary. Unlike them, Bessel oscillations turn out to be localized for the most part inside the vortex because the disturbance amplitude outside the vortex appears to be of $O(\omega')$ relative to the amplitude of boundary displacement. This is connected with the fact that according to (2.3) the phase velocity of the vortex boundary disturbances for Bessel oscillations appears to be close to the phase velocity of the main flow, i.e. it is as if these disturbances *adhere* to the main flow and only faintly disturb the external region.

This means that from the standpoint of sound radiation by a cylindrical vortex the most interesting modes are isolated ones. For a vortex ring, as will be seen below, the situation abruptly changes.

At $k = 0$ the isolated mode reduces to the well-known oscillations of a plane circular vortex (Lamb 1932) with the frequency $\omega = (m-1)/2$. The frequencies of all Bessel modes turn out to equal $\omega = m/2$, i.e. these modes degenerate. In this case any vortex core disturbance of the type $\Omega^{\rho} = 0$, $\Omega^{\phi} = 0$,

$$\Omega^{s} = \left[f(\rho) - \delta(\rho - 1) \int_0^1 f(\rho)\, \rho^{m+1}\, \mathrm{d}\rho \right] \mathrm{e}^{\mathrm{i}m\phi},$$

where $f(\rho)$ is an arbitrary function, will represent an eigen-oscillation with the frequency $\omega = m/2$. Despite the fact that the vorticity disturbance in the core (the first term) is accompanied by a boundary displacement with an amplitude of $O(1)$ (the term with the δ-function), such oscillations produce no disturbance at all in the external region ($\rho > 1$) since their phase velocity exactly equals $1/2$, i.e. these disturbances exactly adhere to the main flow. The set of these modes and of the isolated modes is the complete basis of disturbances in which any two-dimensional initial vortex core disturbance can be expanded. Owing to the triviality of the degenerate modes in the case of cylindrical vortex oscillations, these modes are usually not discussed.

The solutions with $m \leqslant -1$ are easily obtained from the solutions with $m \geqslant 1$ with the help of the operation of complex conjugation of the amplitudes $v^{i}(\rho)$ and simultaneous substitution of $v^{s} \rightarrow -v^{s}$ and of $m \rightarrow -m$, $\omega \rightarrow -\omega$ in the amplitudes $v^{i}(\rho)$.

Considering the whole set of eigen-oscillations with various values of m, it is easy to see that the eigen-frequencies of the cylindrical vortex are localized only near the values $\omega = l/2$ ($l = 0, \pm 1, \pm 2, \ldots$). Modes of two types are available about any such value: an infinite family of Bessel modes which have the form of the harmonic $\exp(\mathrm{i}m\phi)$ with $m = l$ and the single isolated mode which has the form of the harmonic with the next number $m = l+1$ (figure 2).

As will be shown below, the vortex ring oscillations have a similar spectrum. All the eigen-frequencies are localized near $\omega = l/2$ and the set of eigen-oscillations is divided into two families which by analogy with the cylindrical vortex will be called Bessel modes and isolated modes.

Note that the cylindrical vortex eigen-oscillations with a given value $k \ll 1$ can be classified both by the number m of the harmonic $\exp(\mathrm{i}m\phi)$ ((2.3)–(2.5)) and by the number l characterizing the eigen-frequency $\omega \approx l/2$ (figure 2). There is only the second possibility for the vortex ring, since each vortex ring eigen-oscillation has the form of the sum of the harmonics $\exp(\mathrm{i}m\phi)$ with different m, unlike the cylindrical vortex eigen-oscillation which has the form of a separate harmonic in ϕ.

3. A vortex ring with a uniform vorticity distribution and its axisymmetric oscillations

The vortex ring is a localized vortex characterized by a self-induced movement with constant velocity along its axis. In this section we consider the simplest vorticity distribution over the core cross-section, assuming that the magnitude of the vorticity is proportional to the distance to the ring symmetry axis. For such a vorticity distribution there is a steady state in the coordinate system moving with the ring. Note that the steady flow exists not only for rings with thin cores (Fraenkel 1970) but also for thick rings (Norbury 1973) up to the limiting state corresponding to Hill's vortex (Milne-Thompson 1960). In the present work only thin rings ($\mu \ll 1$) are considered.

This stationary flow appears to be the most convenient for consideration of axisymmetric oscillations of the vortex ring, because the displacement of the vortex boundary produces no vorticity disturbances inside the core in this case. The isolated axisymmetrical oscillations of a vortex ring with uniform vorticity reduce to the vortex boundary disturbances only. These oscillations have been discussed previously. Thus Moore (1980) found a solution which was a generalization of a cylindrical vortex with an elliptical section (Kirchhoff's vortex) for the case of a vortex ring. This solution with small eccentricity corresponds to that of the axisymmetric modes. The complete family of isolated axisymmetric modes and the sound field for the main sound-generating mode were obtained by Kopiev & Leontiev (1987).

Isolated oscillations seem not to comprise the whole set of axisymmetrical disturbances of the vortex ring. Like the case of a vortex column (§2) there exists a set of vortex ring oscillations describing the vorticity disturbance inside the core. However, unlike the vortex column oscillations, the frequencies of which degenerate to the point $\omega = m/2$, the corresponding oscillations for the vortex ring with uniform vorticity occupy the frequency range of $\omega/l = (V^{\psi}_{0min}, V^{\psi}_{0max})$. They are the disturbances of the continuous spectrum.

The appearance of the continuous spectrum is associated with non-isochronism of the vortex ring with uniform vorticity because the period of fluid particle rotation over steady flow streamlines in the core section is not a constant (§3.1). This seems to be surprising because the flow in a vortex column (the two-dimensional analogue of a vortex ring with uniform vorticity distribution) is obviously isochronous. However, the non-isochronism of fluid particle orbits can be readily seen in an example of the limiting case – Hill's vortex, for which the period of rotation tends to infinity on approaching the vortex boundary. The continuous spectrum disturbances are considered in §3.3.

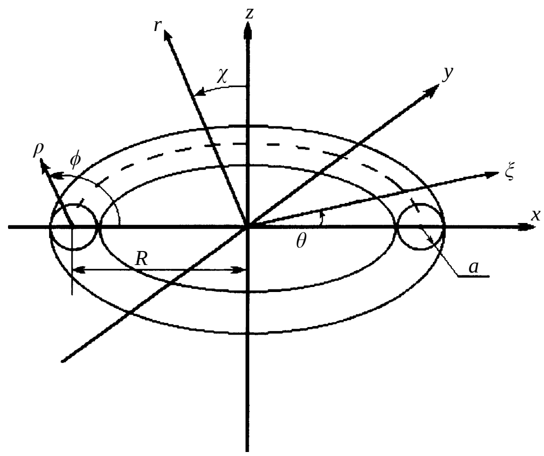

FIGURE 3. Systems of coordinates: x, y, z, Cartesian; z, ξ, θ, cylindrical; r, θ, χ, spherical; ρ, ϕ, polar (in the core section); $a/R = \mu$.

3.1. *Steady flow and coordinate system σ, ψ, s*

Consider the cylindrical coordinates ξ, θ, z with the axis $\boldsymbol{e}_z$ along the ring axis and the polar coordinates ρ, ϕ in the core cross-section with the centre in the stagnation point (figure 3). These coordinates are connected by the relations $\xi = R - \rho\cos\phi$ and $z = \rho\sin\phi$, where R is the distance between the ring axis and the stagnation point. We also use the coordinate s, connected to the angular coordinate θ by the equation $s = R\theta$.

For a uniform vorticity distribution the magnitude of the vorticity is proportional to the distance to the ring symmetry axis, that is $\boldsymbol{\Omega}_0 = \boldsymbol{e}_s \Omega_0 \xi/R$ ($\Omega_0 = \text{const}$). The contravariant s-component of this vorticity field is $\Omega_0^s = \Omega_0$. The steady velocity field $\boldsymbol{V}_0$ can be written as $\boldsymbol{V}_0 = \boldsymbol{V}_\infty + \boldsymbol{U}_0$ where $\boldsymbol{V}_\infty$ is the flow velocity at infinity; the velocity field $\boldsymbol{U}_0$ decreases to zero at infinity and is determined according to the Biot–Savart law:

$$\boldsymbol{U}_0 = \boldsymbol{\nabla} \times \frac{1}{4\pi}\int \frac{\boldsymbol{\Omega}_0(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\mathrm{d}\boldsymbol{r}'.$$

For a thin vortex ring the problem includes the small parameter μ which is the ratio of the core cross-section dimension a to the ring radius R, $\mu = a/R$. Since the core boundary is close to circular and differs from it by terms of $O(\mu^2)$, the value a is determined as the radius of the circle with area πa^2, exactly equal to the core section area.

The contravariant components of the steady velocity field inside the vortex ring core and the core boundary shape are as follows:

$$V_0^\rho = -\mu\Omega_0\frac{5\rho^2}{16a}\sin\phi + \mu^2\Omega_0\left[\left(\frac{3}{8}\ln\frac{8}{\mu} - \frac{15}{64}\right)\rho - \frac{\rho^3}{16a^2}\right]\sin 2\phi + O(\mu^3), \tag{3.1a}$$

$$V_0^\phi = \frac{\Omega_0}{2} + \mu^2\Omega_0\frac{5}{16} - \mu\Omega_0\frac{7\rho}{16a}\cos\phi + \mu^2\Omega_0\left[\left(\frac{3}{8}\ln\frac{8}{\mu} - \frac{15}{64}\right) - \frac{\rho^2}{32a^2}\right]\cos 2\phi + O(\mu^3), \tag{3.1b}$$

$$\rho = a\left[1 + \frac{5}{8}\mu\cos\phi - \frac{25}{256}\mu^2 + \left(\frac{161}{256} - \frac{3}{8}\ln\frac{8}{\mu}\right)\mu^2\cos 2\phi\right] + O(\mu^3), \tag{3.2}$$

where, for brevity, the expression $O(\mu^n)$ is used to indicate terms of order $\mu^n\ln\mu$ and μ^n.

Note that for a description of the stationary flow in a vortex ring the other polar coordinates ρ_c, ϕ_c can be also used with the origin at the ring cross-section centre which is closer to the ring axis than the stagnation point by the distance $\Delta\xi = \frac{5}{8}\mu + O(\mu^3)$ (Fraenkel 1972). Equations (3.1), (3.2) expressed in coordinates ρ_c, ϕ_c at $a = 1$ exactly coincide with that used by Widnall & Tsai (1977). In particular, in these coordinates the boundary shape (3.2) is rather simple and is expressed as

$$\rho_c = a\left[1+\mu^2\left(-\frac{3}{8}\ln\frac{8}{\mu}+\frac{17}{32}\right)\cos 2\phi_c + O(\mu^3)\right].$$

Further, we define the coordinates σ, ψ in the core cross-section of the ring, which slightly differ from the coordinates ρ, ϕ, respectively. We require that for the coordinates $\sigma(\rho,\phi)$ and $\psi(\rho,\phi)$ the following conditions be fulfilled:

$$V_0^\sigma = 0, \quad V_0^\psi = V_0^\psi(\sigma), \quad |\boldsymbol{g}|^{1/2} = \sigma, \tag{3.3}$$

where V_0^σ, V_0^ψ are the contravariant components of the velocity field, g_{ij} is the metric tensor of the coordinate system σ, ψ, s.

The lines $\sigma = \text{const}$ correspond to the streamlines because $V_0^\sigma = 0$. Note that the coordinate transformation $\rho, \phi \to \sigma, \psi$ is different from the Fraenkel (1970) transformation of the type $\rho, \phi \to f(\rho,\phi), \phi$, where the function f also appears to be constant on the streamlines. The present work additionally uses transformation of the angular variable $\phi \to \psi(\rho,\phi)$ which allows consistency of the component V_0^ψ on the streamlines to be obtained. The condition $|\boldsymbol{g}|^{1/2} = \sigma$ is chosen because it gives the simplest forms of differential operators in the coordinate system σ, ψ, s.

In the following equations dimensionless variables are used. The time scale is Ω_0^{-1}. The length scale is selected so that the vortex boundary (3.2) corresponds to the line $\sigma = 1$: just the length scale $a[1+\frac{5}{16}\mu^2 + O(\mu^3)]$ meets this condition.

Making use of the equations

$$V_0^\sigma = \frac{\partial\sigma}{\partial\rho}V_0^\rho + \frac{\partial\sigma}{\partial\phi}V_0^\phi, \quad V_0^\psi = \frac{\partial\psi}{\partial\rho}V_0^\rho + \frac{\partial\psi}{\partial\phi}V_0^\phi, \quad |\boldsymbol{g}(\rho,\phi,s)|^{1/2} = \frac{\partial(\sigma,\psi)}{\partial(\rho,\phi)}|\boldsymbol{g}(\sigma,\psi,s)|^{1/2}$$

where V_0^ρ and V_0^ϕ are determined by (3.1), $|\boldsymbol{g}(\rho,\phi,s)|^{1/2} = \rho(1-\mu\rho\cos\phi)$, we get

$$\sigma = \rho - \mu\frac{5}{8}\rho^2\cos\phi + \mu^2\left[\left(\frac{3}{8}\ln\frac{8}{\mu}-\frac{15}{64}\right)\rho - \frac{1}{256}\rho^3\right]\cos 2\phi + \mu^2\frac{45}{256}\rho^3 + O(\mu^3), \tag{3.4a}$$

$$\psi = \phi + \mu\frac{7}{8}\rho\sin\phi + \mu^2\left[\left(-\frac{3}{8}\ln\frac{8}{\mu}+\frac{15}{64}\right)+\frac{11}{128}\rho^2\right]\sin 2\phi + O(\mu^3). \tag{3.4b}$$

The surface $\sigma = 1$ corresponds to the vorticity region boundary. The covariant components of the normal vector to this boundary are expressed as:

$$\boldsymbol{n} = (n_\sigma, n_\psi, n_s) = (1+\tfrac{5}{4}\mu\cos\psi + O(\mu^2), 0, 0). \tag{3.5}$$

The metrical tensor of the coordinate system σ, ψ, s is presented in Appendix A. The components of the steady velocity field (3.1) in these coordinates are as follows:

$$V_0^\sigma = 0, \quad V_0^\psi = \tfrac{1}{2} - \mu^2\frac{21}{64}\sigma^2 + O(\mu^3). \tag{3.6}$$

This equation demonstrates that, in contrast to the cylindrical vortex with constant vorticity, the vortex ring with uniform vorticity turns out to be the non-isochronous,

(a) (b)

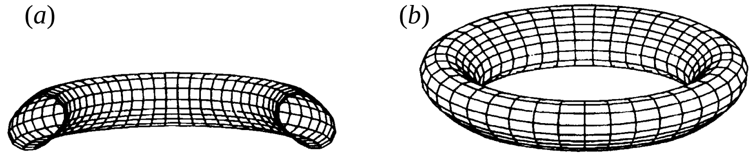

FIGURE 4. Boundary disturbance shape of axisymmetric modes at different phases (isolated mode and continuous spectrum modes), $l = 1$: (a) half a ring; (b) the whole ring.

i.e. the time taken for fluid particle circulation in the vortex core on different streamlines $\sigma = \text{const}$ appears to be different. Now calculate the period of fluid particle rotation on the streamlines: this is

$$T = \oint \frac{\mathrm{d}l}{|\boldsymbol{V}_0|_{\sigma,\, s=\text{const}}},$$

where $\mathrm{d}l = (g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j)^{1/2}$. Along the streamlines the length element $\mathrm{d}l = g_{\psi\psi}^{1/2}\,\mathrm{d}\psi$ and the velocity field module $|\boldsymbol{V}_0| = (g_{ij}\,V_0^i\,V_0^j)^{1/2} = g_{\psi\psi}^{1/2}\,V_0^\psi$. The result is

$$T = 2\pi / V_0^\psi(\sigma). \tag{3.7}$$

Since for the flow examined V_0^ψ is function of σ, the rotation period is different on different streamlines. However, this difference is of $O(\mu^2)$, that is the vortex ring with uniform vorticity differs slightly from isochronous flow.

3.2. *Isolated axisymmetric oscillations*

Consider first the isolated modes. The isolated axisymmetric oscillations of the vortex ring (independent of θ) are the simplest because the problem in this case appears to be two-dimensional. Such disturbances produce no vorticity inside the core and reduce to the boundary displacement only. For these oscillations the vortex boundary displacement $\sigma = 1 + f(\psi)\exp(-\mathrm{i}\omega t)$ has the following form:

$$f = \mathrm{e}^{\mathrm{i}(l+1)\psi} - \mu\frac{2l+3}{4(l+1)}\mathrm{e}^{\mathrm{i}(l+2)\psi} + \mu\frac{2l+1}{4(l+1)}\mathrm{e}^{\mathrm{i}l\psi} + O(\mu^2), \quad l = 1, 2, \ldots. \tag{3.8}$$

Thus, these oscillations, in the leading approximation, are $(l+1)$-fold displacements of the vortex boundary (figure 4). The eigen-frequency is

$$\omega_0 = \frac{l}{2} + \mu^2\left[\frac{(6l^3+18l^2+14l+3)}{16l(l+1)(l+2)} - \frac{21}{64}(l+1)\right] + O(\mu^3). \tag{3.9}$$

3.3. *Continuous spectrum of axisymmetric oscillations*

To obtain the complete basis of oscillations of the form $\exp(-\mathrm{i}\omega t)$ in which any initial vortex core disturbance can be expanded it is necessary to find the oscillations which, contrary to the isolated ones, are accompanied by vorticity disturbances inside the vortex core. The linearized Helmholtz equation describing the vorticity disturbances is

$$-\mathrm{i}\omega\Omega^s + V_0^\psi\frac{\partial\Omega^s}{\partial\psi} + v^\sigma\frac{\partial\Omega_0^s}{\partial\sigma} = 0, \tag{3.10}$$

where v^σ is the velocity disturbance connected with the vorticity disturbance by the Biot–Savart law, V_0^ψ is determined by (3.3), $\Omega_0^s = 1$ at $\sigma \leqslant 1$, $\Omega_0^s = 0$ at $\sigma > 1$; Ω^s is the only vorticity disturbance component that is not zero.

FIGURE 5. The isolated mode frequency (○) and continuous spectrum for axisymmetric disturbances.

The solution is found in the form

$$\Omega^s = \delta(\sigma-\sigma_0)\,\mathrm{e}^{\mathrm{i}l\psi} + a(\psi)\,\delta(\sigma-1), \quad l = 1,2,\ldots, \tag{3.11}$$

where $\sigma_0 < 1$. The first term in this expression is the vorticity disturbance of unit amplitude inside the core on the streamline $\sigma = \sigma_0$. The second term is the unknown displacement of the vortex boundary, corresponding to this core disturbance. Substituting (3.11) into (3.10) and equating to zero the multipliers of two δ-functions we get the equations

$$-\omega + lV_0^\psi(\sigma_0) = 0, \tag{3.12}$$

$$-\mathrm{i}\omega a + V_0^\psi(1)\frac{\partial a}{\partial \psi} - v_2^\sigma|_{\sigma=1} = v_1^\sigma|_{\sigma=1}, \tag{3.13}$$

where $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are the velocity fields produced at the boundary $\sigma = 1$ by the first and the second terms of the vorticity (3.11) according to Biot–Savart's law. Equation (3.12) determines eigen-frequencies which depend on σ_0,

$$\omega = lV_0^\psi(\sigma_0). \tag{3.14}$$

The frequencies of these oscillations are within the interval $\omega/l = (V_{0\,\mathrm{min}}^\psi, V_{0\,\mathrm{max}}^\psi)$ and correspond to the continuous spectrum, the appearance of which is connected with the non-isochronous character of the vortex ring with a uniform vorticity profile. The location of the continuous and discrete spectra is presented on figure 5.

Equation (3.13) determines the response amplitude $a(\psi)$ of the vortex boundary to the disturbance produced by the δ-like vorticity located on the layer $\sigma = \sigma_0$. If the right-hand side of the equation equals zero, this equation describes the vortex boundary disturbance for isolated oscillations (3.8). Since the right-hand side is a sum of different harmonics with frequency close to the eigen-frequency (3.9), the response is of resonance character. Making use of Appendix B we easily get

$$a(\psi) = C[\mathrm{e}^{\mathrm{i}(l+1)\psi} + \mu a_{l+2}\,\mathrm{e}^{\mathrm{i}(l+2)\psi} + \mu a_l\,\mathrm{e}^{\mathrm{i}l\psi} + O(\mu^2)], \tag{3.15a}$$

$$a_{l+2} = -\frac{2l+3}{4(l+1)}, \quad a_l = \frac{2l+1}{4(l+1)} - \frac{\sigma_0^{l+1}}{\mu C}, \tag{3.15b}$$

$$C = \mu\frac{(3l+2)}{8}\frac{\sigma_0^{l+1}(1-\sigma_0^2)}{\omega-\omega_0}, \tag{3.15c}$$

where ω_0 and ω are determined by (3.9) and (3.14) respectively. In the case of a δ-like vorticity on the critical layer ($\omega = \omega_0$), the amplitude of the boundary displacement becomes infinite.

It is surprising that it is not the harmonic $\exp(\mathrm{i}l\psi)$ determining the form of vorticity disturbances inside the core that has the maximum amplitude in the function $a(\psi)$, but the harmonic $\exp[\mathrm{i}(l+1)\psi]$. Thus the oscillations (3.15), in the leading approximation represent not the lth harmonic but the $(l+1)$th one. The phase velocity of $(l+1)$th harmonic does not coincide with the flow velocity. Therefore, all the modes of the continuous spectrum of the vortex ring disturb the external field and consequently can participate in the process of noise generation. We see the dramatic difference between these oscillations and the corresponding oscillations of the cylindrical vortex at $k = 0$ (§2) which have the form of the harmonic $\exp(\mathrm{i}l\psi)$ and produce no disturbances in the external field ($\rho > 1$) because they exactly adhere to the main flow.

Note that this feature associated with appearance of harmonics not adhered to the main flow is also observed for three-dimensional modes of the vortex ring considered below and this is their principal difference from the cylindrical vortex oscillations, the majority of which are adherent, to the main approximation, to the steady flow and slightly disturb the external region.

Finally in this section, the normalized boundary displacement of the axisymmetrical modes are written as

$$f = \mathrm{e}^{\mathrm{i}(l+1)\psi} - \mu\frac{2l+3}{4(l+1)}\mathrm{e}^{\mathrm{i}(l+2)\psi} + \mu\left[\frac{2l+1}{4(l+1)} - \frac{8(\omega-\omega_0)}{\mu^2(3l+2)(1-\sigma_0^2)}\right]\mathrm{e}^{\mathrm{i}l\psi} + O(\mu^2). \quad (3.16)$$

The amplitude of the $(l+1)$th harmonic in (3.16) is assumed to be unity for each mode.

4. The problem of three-dimensional vortex ring oscillations

Two main difficulties appear in the problem of three-dimensional oscillations of a vortex ring. The first is associated with the selection of the steady flow. It was shown in §3.1 that a vortex ring with a uniform distribution of vorticity is not isochronous. The non-isochronism of fluid particles leads to the appearance of continuous spectrum disturbances. The investigation of the continuous spectrum is a rather complex task. Though two-dimensional (axisymmetric) disturbances of the continuous spectrum were obtained in an analytical form in §3.3, consideration of the continuous spectrum in a general case is outside the scope of this paper. Therefore, we investigate the three-dimensional oscillations not of the vortex ring with uniform vorticity but of the isochronous vortex ring for which the oscillation spectrum turns out to be purely discrete. Such a vortex ring is constructed in §4.1.

The second difficulty is associated with the fact that for three-dimensional eigen-oscillations it is unknown beforehand in what form the solution is to be found, although it is easy to write this form in the cases of vortex column oscillations (§2) or vortex ring two-dimensional (axisymmetric) oscillations (§§3.2 and 3.3). In particular, the use of vortex column modes as the main approximation for three-dimensional vortex ring modes appears to be unsatisfactory. Therefore, in the present work a method based on the expansion of eigen-oscillations over some specially constructed set of the so-called basic disturbances (§4.4) is developed. In this procedure the coefficients of the eigen-oscillation expansion in basic disturbances and eigen-frequency are simultaneously found from the boundary condition. It appears to be convenient for this purpose to examine the task in terms of the displacement field $\boldsymbol{\varepsilon}$ (§4.2).

4.1. *A steady vortex ring with isochronous movement of fluid particles*

A vortex ring with a uniform vorticity distribution appears to be isochronous only for two approximations in μ (§3.1). The isochronous condition is violated for the terms of $O(\mu^2)$. In a general case, to find the vorticity distribution satisfying the isochronous condition is a complex task. However, for terms of $O(\mu^2)$ it is easy to modify the vorticity distribution considered in §3.1 to meet this condition.

It is known that for any vorticity distribution with $\Omega_0^s = \text{const}$ on the streamlines there exists a steady thin vortex ring (Fraenkel 1970, theorem 3.1). Let us consider a vorticity distribution slightly different from the uniform one, $\Omega_0^s = 1 + \Delta\Omega$, where $\Delta\Omega$ is an arbitrary function of $O(\mu^2)$, which is a constant on the streamlines. Obviously, in the leading approximation this function has the form $\Delta\Omega = \mu^2 F(\rho) + O(\mu^3)$. We shall demonstrate now that for such a vorticity distribution the coordinates σ and ψ found from the condition (3.3) will differ from the coordinates (3.4) found for the uniform vorticity distribution only by an amount of $O(\mu^3)$. According to Fraenkel's theorem, a steady vortex ring does in fact exist with such vorticity distribution. This vorticity field corresponds to the steady velocity field which differs from (3.1) by $\Delta V_0^\rho = O(\mu^3)$, $\Delta V_0^\phi = \mu^2 g(\rho) + O(\mu^3)$, where $\mathrm{d}g/\mathrm{d}\rho = -F(\rho)$. This means that (i) the streamlines of the flow examined will differ from the streamlines of the flow with a uniform vorticity distribution by an amount of $O(\mu^3)$, (ii) ΔV_0^ϕ, in the leading approximation, is independent of the coordinate ϕ. Therefore the coordinates complying with the conditions (3.3) for different F differ in terms of $O(\mu^3)$ only, i.e. are determined by the expressions (3.4). Thus, the steady vorticity field in question has the form $\Omega_0^s = 1 + \mu^2 F(\sigma)$.

The function $F(\sigma)$ is selected in such a way that the component V_0^ψ becomes constant on all the streamlines. The function $F(\sigma) = \frac{21}{16}\sigma^2$ corresponds to this condition. In this case the steady fields of vorticity and velocity are as follows:

$$\Omega_0^s = 1 + \mu^2 \tfrac{21}{16}\sigma^2 + O(\mu^3), \quad V^\sigma = 0, \quad V^\psi = \tfrac{1}{2}, \tag{4.1}$$

where coordinates σ, ψ are determined by (3.4).

According to (3.7) this flow will be isochronous in the vortex core. This flow is used in the present work for the investigation of three-dimensional oscillations.

4.2. *Displacement field in the problem of vortex flow disturbances*

The approach to a description of three-dimensional vortex ring linear disturbances in this paper is based on using the displacement field $\boldsymbol{\varepsilon}(\boldsymbol{r})$ as the main function (Chandrasekhar 1969; Drazin & Reid 1981). A small displacement of the fluid elements from the points $\boldsymbol{r}$ to the points $\boldsymbol{r} + \boldsymbol{\varepsilon}(\boldsymbol{r})$, $\nabla \cdot \boldsymbol{\varepsilon} = 0$, gives the vorticity disturbance:

$$\boldsymbol{\Omega} = \nabla \times (\boldsymbol{\varepsilon} \times \boldsymbol{\Omega}_0), \tag{4.2}$$

where $\boldsymbol{\Omega}_0$ is the steady vorticity field. Equation (4.2) is the condition of the vortex line freezing into the fluid element (Moffatt 1986).

The system of equations describing the displacement field evolution is

$$\frac{\partial \boldsymbol{\varepsilon}}{\partial t} + \nabla \times (\boldsymbol{\varepsilon} \times \boldsymbol{V}_0) - \boldsymbol{v} = 0, \tag{4.3a}$$

$$\boldsymbol{v}(\boldsymbol{r}) = \nabla \times \frac{1}{4\pi} \int \frac{\nabla' \times (\boldsymbol{\varepsilon} \times \boldsymbol{\Omega}_0)}{|\boldsymbol{r} - \boldsymbol{r}'|} \mathrm{d}\boldsymbol{r}', \tag{4.3b}$$

where $\nabla' = \partial/\partial \boldsymbol{r}'$, $\boldsymbol{V}_0$ is the steady velocity field, $\boldsymbol{v}$ is the velocity disturbance. Equation (4.3*a*) was first obtained by Drazin & Reid (1981). Equation (4.3*b*) is obtained by substitution of (4.2) into the Biot–Savart law.

Let us consider the situation when vorticity is concentrated in a localized region of the flow. In this case the system (4.3) is reduced to the form which allows the construction of an efficient procedure for the vortex eigen-oscillation calculation.

It is well-known (Batchelor 1967) that the vector field $\boldsymbol{a}$ in a localized region M can be determined from the normal component of the field $\boldsymbol{a}$ at the boundary $\Gamma(M)$ and from curl $\boldsymbol{a}$ and div $\boldsymbol{a}$ over the whole region M.

Note that if the region is not simply connected, in order to determine the vector field $\boldsymbol{a}$ unambiguously it is also necessary to specify the circulation of that field around a closed contour C not reducible to zero. But this condition is trivial for vortex ring oscillations of the kind of $\exp(\mathrm{i}n\theta)$.

Let the expression $\partial\boldsymbol{\varepsilon}/\partial t+\boldsymbol{\nabla}\times(\boldsymbol{\varepsilon}\times \boldsymbol{V}_0)-\boldsymbol{v}$ be $\boldsymbol{a}$ and the vortex region be M. Then it follows that (4.3*a*) is equivalent to the system of equations

$$\frac{\partial}{\partial t}\boldsymbol{\nabla}\times\boldsymbol{\varepsilon}+\boldsymbol{\nabla}\times[\boldsymbol{\nabla}\times(\boldsymbol{\varepsilon}\times \boldsymbol{V}_0)]-\boldsymbol{\nabla}\times(\boldsymbol{\varepsilon}\times\boldsymbol{\Omega}_0)=0, \quad \boldsymbol{r}\in M, \tag{4.4a}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{\varepsilon}=0, \quad \boldsymbol{r}\in M, \tag{4.4b}$$

$$\left(\frac{\partial\boldsymbol{\varepsilon}}{\partial t}+\boldsymbol{\nabla}\times(\boldsymbol{\varepsilon}\times \boldsymbol{V}_0)-\boldsymbol{v}\right)\cdot\boldsymbol{n}=0, \quad \boldsymbol{r}\in G(M), \tag{4.4c}$$

where $\boldsymbol{n}$ is the normal to the surface $\Gamma(M)$. An important advantage of the system (4.4) in comparison with (4.3*a*) is that the integral term $\boldsymbol{v}$ is excluded from the equation in region M and it is to be evaluated only at the vortex boundary $\Gamma(M)$.

Using an integration by parts for the case of a localized vortex we obtain

$$\boldsymbol{v}=\boldsymbol{\varepsilon}\times\boldsymbol{\Omega}_0+\boldsymbol{\nabla}\frac{1}{4\pi}\int_M\frac{\boldsymbol{\nabla}'[\boldsymbol{\varepsilon}\times\boldsymbol{\Omega}_0]}{|\boldsymbol{r}-\boldsymbol{r}'|}\mathrm{d}\boldsymbol{r}'. \tag{4.5}$$

In the region outside the vortex $\boldsymbol{\varepsilon}\times\boldsymbol{\Omega}_0=0$, therefore the external velocity disturbance $\boldsymbol{v}$ can be presented in accordance with (4.5) as the field generated by a source with the density $Q(\boldsymbol{r})=-\boldsymbol{\nabla}[\boldsymbol{\varepsilon}\times\boldsymbol{\Omega}_0]$, that is

$$\boldsymbol{v}=\boldsymbol{\nabla}\Phi, \quad \Phi=-\frac{1}{4\pi}\int_M\frac{Q(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\mathrm{d}\boldsymbol{r}'. \tag{4.6}$$

For the calculation of $(\boldsymbol{v}\cdot\boldsymbol{n})$ in (4.4*c*) both external and internal limits of the expression (4.5) can be used since the normal component of velocity $\boldsymbol{v}$ is continuous at the vortex boundary $\Gamma(M)$; however, we use the external limit, (4.6).

It is to be noted that the density Q in (4.6) can be expressed in more than one way. The potential Φ outside region M is not changed when Q is replaced by the expression

$$Q=-\boldsymbol{\nabla}\cdot[(\boldsymbol{\varepsilon}\times\boldsymbol{\Omega}_0)+\boldsymbol{\nabla}G], \tag{4.7}$$

where G is an arbitrary function different from zero only in the region occupied by the vortex. This ambiguity in the selection of Q is used below for the reduction of the volume integral (4.6) to the surface one.

Now we consider a physical aspect of the system of equations (4.4), (4.6). Equation (4.4*c*) describes the evolution of the disturbed vortex boundary. Equations (4.4*a*), (4.4*b*) describe the evolution of disturbances inside the vortex. The vortex boundary evolution is connected with the disturbances inside the vortex through the term $(\boldsymbol{v}\cdot\boldsymbol{n})$ which is expressed as an integral over the whole volume of the vortex. In turn, the dependence of the internal disturbances on the form of the boundary is expressed in the fact that (4.4*c*) serves as the boundary condition for (4.4*a*), (4.4*b*). Equation (4.6) takes into account the behaviour of the velocity at infinity.

Thus the system of equations (4.4), (4.6) describes disturbances of a three-dimensional vortex in terms of the joint evolution of the vortex boundary and disturbances inside the vortex region. Such an approach can be treated as a generalization of the method of contour dynamics applied to describe the evolution of a two-dimensional vortex boundary with constant vorticity (Zabusky, Hughes & Roberts 1979; Dritchel 1986) to the case of the disturbances of three-dimensional localized vortices with arbitrary vorticity.

4.3. *Governing equations*

The vortex ring symmetry with respect to rotation around its axis z permits one to look for eigen-oscillations in the form

$$\epsilon^i(\boldsymbol{r}, t) = \epsilon^i(\sigma, \psi)\exp(\mathrm{i}n\theta - \mathrm{i}\omega t). \tag{4.8}$$

The amplitude function $\epsilon^i(\sigma, \psi)$ has to be found in order to determine the disturbance forms in a vortex ring. This function is a vector function of two variables, σ and ψ. For this reason the problem of vortex ring oscillations is much more difficult than the problem of cylindrical vortex oscillations which reduces to finding the functions of only one variable, ρ.

The coordinates σ, ψ, s are a curvilinear coordinate system with the non-diagonal metric tensor g_{ij} (Appendix A). Making use of such system one must keep in mind the difference between co- and contravariant components of the vector fields $a_i = g_{ij}\,a^j$. In the coordinate system σ, ψ, s (4.2) is as follows:

$$\Omega^i = \mathrm{i}\mu n\Omega_0(\sigma)\,\epsilon^i + \left[\delta(\sigma-1)\,\Omega_0(\sigma) - \frac{\mathrm{d}\Omega_0}{\mathrm{d}\sigma}\right]\delta_{is}\,\epsilon^\sigma, \tag{4.9}$$

where δ_{ij} is the Kronecker symbol, $\delta(x)$ is Dirac's δ-function. From (4.9) it follows that at $n \neq 0$ the variables $\boldsymbol{\Omega}$ and $\boldsymbol{\varepsilon}$ are interconnected unambiguously for all n.

The system of equations (4.4) in the coordinates σ, ψ, s at $\sigma < 1$ takes the form

$$\frac{\partial}{\partial\psi}g_{ss}\,\mathrm{L}\epsilon^s - \mathrm{i}\mu n(g_{\psi\psi}\,\mathrm{L}\epsilon^\psi + g_{\psi\sigma}\,\mathrm{L}\epsilon^\sigma) - \mathrm{i}\mu n\Omega_0\,\sigma\epsilon^\sigma = 0, \tag{4.10a}$$

$$-\frac{\partial}{\partial\sigma}g_{ss}\,\mathrm{L}\epsilon^s + \mathrm{i}\mu n(g_{\sigma\sigma}\,\mathrm{L}\epsilon^\sigma + g_{\sigma\psi}\,\mathrm{L}\epsilon^\psi) - \mathrm{i}\mu n\Omega_0\,\sigma\epsilon^\psi = 0, \tag{4.10b}$$

$$\frac{\partial}{\partial\sigma}(g_{\psi\psi}\,\mathrm{L}\epsilon^\psi + g_{\psi\sigma}\,\mathrm{L}\epsilon^\sigma) - \frac{\partial}{\partial\psi}(g_{\sigma\sigma}\,\mathrm{L}\epsilon^\sigma + g_{\sigma\psi}\,\mathrm{L}\epsilon^\psi) - \mathrm{i}\mu n\Omega_0\,\sigma\epsilon^s - \sigma\epsilon^\sigma\frac{\mathrm{d}\Omega_0}{\mathrm{d}\sigma} = 0, \tag{4.10c}$$

$$\frac{1}{\sigma}\frac{\partial}{\partial\sigma}(\sigma\epsilon^\sigma) + \frac{\partial\epsilon^\psi}{\partial\psi} + \mathrm{i}\mu n\epsilon^s = 0, \tag{4.10d}$$

where g_{ij} is the metric tensor, operator L has the form

$$\mathrm{L} = -\mathrm{i}\omega + V_0^\psi\frac{\partial}{\partial\psi}. \tag{4.11}$$

The boundary condition

$$\mathrm{L}\epsilon^\sigma - v^\sigma = 0, \quad \sigma = 1 \tag{4.12}$$

is obtained from (4.4*c*) taking account of the fact that the normal vector (3.5) to the vortex ring surface has the covariant components $n_\psi = 0$, $n_s = 0$. The component v^σ in (4.12) is determined from (4.6), (4.7).

Note that if V_0^ψ depended on σ, the solution of (4.10) would have a singularity on some streamlines (critical layers) $\sigma = \sigma_m$ where $-\omega + mV_0^\psi(\sigma_m) = 0$, m being an integer. This would mean that the problem includes not only discrete eigen-frequencies but also a continuous spectrum. For a thin vortex ring with a uniform vorticity distribution the velocity has the form $V_0^\psi = \frac{1}{2} + \mu^2 F(\sigma)$, i.e. the continuous spectrum appears at higher approximations with respect to μ. The vortex ring with an isochronous flow in the core which is examined in this section differs from all the others in that $V_0^\psi = \frac{1}{2}$ does not depend on σ to any approximation and hence it is characterized by oscillations of the discrete spectrum only. In this respect, the isochronous ring, and not the uniform one, is the simplest for investigating three-dimensional oscillations.

The system of equations (4.10), (4.12) relating to the field $\boldsymbol{\varepsilon}$ and (4.6), (4.7) connecting the fields $\boldsymbol{v}$ and $\boldsymbol{\varepsilon}$ are the governing equations for determining three-dimensional eigen-oscillations. These equations have non-trivial solutions only for certain values of ω which are the eigen-frequencies.

Note that the coefficients of these equations do not change with a change of the sign of the s-component of the field $\boldsymbol{\varepsilon}(\sigma, \psi)$ with a simultaneous replacement of $n \to -n$. Therefore the eigen-oscillations at $n \leqslant -1$ can readily be obtained from the oscillations at $n \geqslant 1$ through the replacement $\epsilon^s \to -\epsilon^s$ and the replacement of $n \to -n$ in the amplitude $\boldsymbol{\varepsilon}(\sigma, \psi)$ in (4.8). Thus only the case $n \geqslant 1$ will be considered further.

4.4. *Method of solution of the problem for $\mu \ll 1$*

In the present work we consider only long-wave oscillations ($n = O(1)$) of a thin ($\mu \ll 1$) vortex ring. The eigen-frequencies of long-wave oscillations of the vortex ring must be located in the regions $\omega = l/2 + O(\mu)$, where l is an integer. In fact, the undisturbed flow in a vortex ring in the limit $\mu \to 0$ transforms to the flow in a cylindrical vortex. Therefore the eigen-frequencies of a thin vortex ring ($\mu \ll 1$) have to differ from eigen-oscillations of a cylindrical vortex ($\mu = 0$) by the value becoming zero at $\mu \to 0$. Since the cylindrical vortex has no long-wave oscillations with the eigen-frequencies outside the regions $\omega \approx l/2$, such oscillations cannot exist for the thin vortex ring either. This conclusion can also be easily found from direct examination of the governing equation set.

There are two complex problems in this task: solution of the system of differential equations (4.10) and calculation of the integral (4.6) which appears in the boundary condition (4.12). In this subsection a method is derived which permits the separation of these problems. First, the eigen-oscillation is presented as an expansion in a set of basic disturbances. Each basic disturbance is a solution of (4.10) but does not satisfy the condition (4.12) at the vortex boundary. Then the velocity field for each basic disturbance is calculated at the vortex boundary using the integral (4.6). Finally, the coefficients of the eigen-oscillation expansion in basic disturbances and the eigen-frequency ω are found from the boundary condition (4.12). The solution of this problem for long-wave oscillations of a thin vortex ring is constructed as a series of successive approximations in the parameter μ.

4.4.1. *Basic disturbances $\boldsymbol{\varepsilon}_{(m)}$*

Now we consider the set of vector fields $\boldsymbol{\varepsilon}_{(m)}(\sigma, \psi)$, $m = 0, \pm 1, \pm 2, \ldots$ in the region $\sigma \leqslant 1$, which are the particular solutions of the system of equations (4.10) where ω is a parameter. We demand that the boundary values of the σ-components of these fields

$$\eta_{(m)}(\psi) = \epsilon^{\sigma}_{(m)|\sigma=1} \tag{4.13}$$

are a complete system of linearly independent functions in the segment $0 \leqslant \psi \leqslant 2\pi$. Then any solution of the system of equations (4.10) at an arbitrary boundary condition (including eigen-oscillations themselves) can be presented as an expansion:

$$\boldsymbol{\varepsilon}(\sigma, \psi) = \sum_{m=-\infty}^{\infty} C_m \, \boldsymbol{\varepsilon}_{(m)}(\sigma, \psi). \tag{4.14}$$

The fields $\boldsymbol{\varepsilon}_{(m)}(\sigma, \psi, \omega)$ are called *basic displacements*. Each basic displacement has the following physical meaning: it describes the internal structure of vortex disturbances at forced oscillations of the vortex boundary in the form $\eta_{(m)}(\psi)$ with the frequency ω.

The basic displacements are selected as the linearly independent particular solutions which have functionally the simplest form. Boundary conditions are not selected beforehand. The completeness and linear independence of the system of functions $\eta_{(m)}(\psi)$ will be confirmed directly by the expressions obtained for the basic displacements.

Consider the method of basic displacement construction in a general case. The basic displacements $\epsilon^i_{(m)}$ and the metric tensor components g_{ij} are presented as Fourier series:

$$\epsilon^i_{(m)}(\sigma,\psi) = \sum_{p=-\infty}^{\infty} \epsilon^i_p(\sigma)\,\mathrm{e}^{\mathrm{i}p\psi}, \tag{4.15}$$

$$g_{ij} = G_{ij} + \sum_{p=-\infty}^{\infty} g^{(p)}_{ij}(\sigma)\,\mathrm{e}^{\mathrm{i}p\psi}. \tag{4.16}$$

The values of $g^{(p)}_{ij}$ are determined in Appendix A and are no greater than $O(\mu)$. Substituting (4.15) and (4.16) into (4.10) we get a system of equations for determining ϵ^i_p, $p = 0,\ \pm 1,\ \pm 2, \ldots$:

$$\mathrm{i}p\mathrm{L}_p\,\epsilon^s_p = \mathrm{i}\mu n(\Omega_0\,\sigma\epsilon^\sigma_p + \sigma^2\,\mathrm{L}_p\,\epsilon^\psi_p) + \sum_{q=-\infty}^{\infty} \mathrm{L}_{p-q}[-\mathrm{i}pg^{(q)}_{ss}\,\epsilon^s_{p-q} + \mathrm{i}\mu n(g^{(q)}_{\psi\psi}\,\epsilon^\psi_{p-q} + g^{(q)}_{\psi\sigma}\,\epsilon^\sigma_{p-q})], \tag{4.17a}$$

$$-\frac{\mathrm{d}}{\mathrm{d}\sigma}\mathrm{L}_p\,\epsilon^s_p = \mathrm{i}\mu n(\mathrm{L}_p\,\epsilon^\sigma_p - \Omega_0\,\sigma\epsilon^\psi_p) + \sum_{q=-\infty}^{\infty} \mathrm{L}_{p-q}\left[\frac{\mathrm{d}}{\mathrm{d}\sigma}(g^{(q)}_{ss}\,\epsilon^s_{p-q}) - \mathrm{i}\mu n(g^{(q)}_{\sigma\sigma}\epsilon^\sigma_{p-q} + g^{(q)}_{\sigma\psi}\,\epsilon^\psi_{p-q})\right], \tag{4.17b}$$

$$\frac{\mathrm{d}}{\mathrm{d}\sigma}(\sigma^2\,\mathrm{L}_p\,\epsilon^\psi_p) - \mathrm{i}p\mathrm{L}_p\,\epsilon^\sigma_p = \mathrm{i}\mu n\Omega_0\,\sigma\epsilon^s_p - \sigma\epsilon^\sigma_p\frac{\mathrm{d}\Omega_0}{\mathrm{d}\sigma}$$
$$+ \sum_{q=-\infty}^{\infty} \mathrm{L}_{p-q}\left[-\frac{\mathrm{d}}{\mathrm{d}\sigma}(g^{(q)}_{\psi\psi}\,\epsilon^\psi_{p-q} + g^{(q)}_{\psi\sigma}\,\epsilon^\sigma_{p-q}) + \mathrm{i}p(g^{(q)}_{\sigma\sigma}\,\epsilon^\sigma_{p-q} + g^{(q)}_{\sigma\psi}\,\epsilon^\psi_{p-q})\right], \tag{4.17c}$$

$$\frac{1}{\sigma}\frac{\mathrm{d}}{\mathrm{d}\sigma}(\sigma\epsilon^\sigma_p) + \mathrm{i}p\epsilon^\psi_p = -\mathrm{i}\mu n\epsilon^s_p, \tag{4.17d}$$

where L_p is the operator L acting on the pth harmonic and having the value $\mathrm{L}_p = \mathrm{i}(p/2-\omega)$. Note that in the notation of the harmonic amplitudes ϵ^i_p there are no brackets in the lower subscript as distinct from the basic displacement $\epsilon^i_{(m)}$.

The harmonic amplitudes $\epsilon^i_p(\sigma)$ are found from (4.17) by successive approximations in the small parameter μ. When considering this method one should take into account the following. If the L_p were of $O(1)$ for every p, the method of successive approximations in respect to the parameter μ would be trivial. In this case the uniform equations (4.17) with right-hand sides equal to zero are solved at the first step. At the next steps the non-uniform equations (4.17) are solved, the right-hand sides of which are expressed through the solutions obtained at the previous step.

However, it was shown at the beginning of this section that long-wave eigen-oscillations of thin vortex ring have frequencies near the values $l/2$, that is

$$\omega = \tfrac{1}{2}l + \omega', \tag{4.18}$$

where $\omega' \ll 1$. It means that L_p at $p = l$ is an additional small parameter of the problem aside from μ. This parameter changes the orders of magnitudes of the terms in (4.17), separating the l-harmonic from all the others. As a result, the method of successive approximations appears to be more complex. Since we confine ourselves to calculation of several expansion terms in the basic displacements, this fact will appear to be significant only for those basic displacements which include the *l*-harmonic to the first or second approximations. A detailed description of the whole mathematical procedure is, however, too complicated to be given in this paper and we write only the resulting equations for the complete set of basic displacements (Appendix C). Obviously, one can add an arbitrary solution of (4.10) to the displacement presented.

4.4.2. *Calculation of the integral term* $V_{(m)}$

The next step of the method consists in calculating $V_{(m)}(\psi)$ for each basic displacement $\boldsymbol{\varepsilon}_{(m)}$. In accordance with (4.6) this is determined from the equation

$$V_{(m)} = v^{\sigma}_{(m)}|_{\sigma=1}, \quad \boldsymbol{v}_{(m)} = -\boldsymbol{\nabla}\frac{1}{4\pi}\int\frac{Q_{(m)}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\mathrm{d}\boldsymbol{r}', \tag{4.19}$$

where $Q_{(m)}$ is found by substitution of the field $\boldsymbol{\varepsilon}_{(m)}$ in (4.7). $V_{(m)}$ characterizes the normal component of the velocity disturbance at the vortex boundary which would be generated by the vorticity disturbance $\boldsymbol{\Omega} = \boldsymbol{\nabla}\times(\boldsymbol{\varepsilon}_{(m)}\times\boldsymbol{\Omega}_0)$ in accordance with the Biot–Savart law.

$Q_{(m)}$ can be selected in more than one way. According to (4.7) the same value of the velocity is produced by sources with different densities $Q_{(m)} = -\boldsymbol{\nabla}\cdot(\boldsymbol{\varepsilon}_{(m)}\times\boldsymbol{\Omega}_0+\boldsymbol{\nabla}G)$ where G is an arbitrary function equal to zero at $\sigma > 1$. An appropriate selection of G can, in principle, help overcome the calculation difficulties. This is connected with the fact that in some cases $Q_{(m)}$ can appear to be a function of σ with an alternating sign. In this case cancellation of terms in the integral (4.19) leads to obtaining the same expression for $\boldsymbol{v}$ with the use of a different number of approximations in the parameter μ for different representations of $Q_{(m)}$.

In this situation the best way forward would be to add such a function of G which reduces the volume density to the surface one, i.e.

$$Q_m(\sigma,\psi) = q_m(\psi)\,\delta(\sigma-1), \tag{4.20}$$

where $\delta(x)$ is Dirac's delta-function. The function G is obtained as a solution of

$$\nabla^2 G = -\boldsymbol{\nabla}\cdot(\boldsymbol{\varepsilon}_{(m)}\times\boldsymbol{\Omega}_0), \quad \sigma < 1, \tag{4.21a}$$

$$G = 0, \quad \sigma = 1. \tag{4.21b}$$

For each basic displacement the functions G are calculated as series in μ. Then we get the expression

$$q_{(m)} = [\boldsymbol{\varepsilon}_{(m)}\times\boldsymbol{\Omega}_0+\boldsymbol{\nabla}G]^{\sigma}|_{\sigma=1}. \tag{4.22}$$

In spite of the appearance of the intermediate step connected with the calculation of the unknown function G this reduction of the volume density to the surface one is a great advantage because the integration over σ in (4.19) becomes trivial. The integration over ψ is performed with the help of formulas presented in Appendix B. Thus for each basic displacement (Appendix C) the functions G, $q_{(m)}$ and $V_{(m)}$ are successively calculated.

4.4.3. *The system of algebraic equations for determining the coefficients* C_m *and eigen-frequencies* ω

Now we use the boundary condition (4.12) in order to find the eigen-frequencies ω and coefficients C_m of the eigen-oscillation expansion (4.14) in the basic displacements $\boldsymbol{\varepsilon}_{(m)}$. Adopt the notation

$$F_{(m)}(\psi) = \mathrm{L}\eta_{(m)} - V_{(m)}, \tag{4.23}$$

where $\eta_{(m)}$ and $V_{(m)}$ are determined from (4.13) and (4.19), respectively. Then substituting (4.23) into (4.12) and using (4.14) we get an infinite system of linear algebraic equations:

$$\sum_{m=-\infty}^{\infty} F_{(m)p} C_m, \quad p = 0, \pm 1, \pm 2, \ldots, \tag{4.24}$$

where $F_{(m)p}$ are the coefficients of the Fourier series of the functions $F_{(m)}(\psi)$ in the harmonics $\exp(\mathrm{i}p\psi)$. This system determines coefficients C_m and eigen-frequencies ω.

The solution of the infinite system of algebraic equations (4.24) in a general case is of significant difficulty. However for a thin vortex ring coefficients $F_{(m)p}$ will have different orders in μ. Therefore for determining coefficients C_m to each approximation the infinite matrix $F_{(m)p}$ will be efficiently reduced to a finite one.

4.5. *Dispersion equation and eigen-frequencies*

The expressions for the auxiliary functions G, $q_{(m)}$, $V_{(m)}$ and $F_{(m)}$ are too cumbersome to be derived in this paper. For this reason only the resulting system of algebraic equations is written here. Consider first the frequencies $\omega \approx l/2$ at $l \geqslant 1$ and then the frequencies $\omega \approx 0$ $(l = 0)$.

In the case $l \geqslant 2$ the equation system (4.24) is as follows:

$$\left[\left(-\frac{l\omega'}{2\mu n} + O(\omega')\right) J_l(a_0) + O(\mu\omega')\, J_{l+1}(a_0)\right] C_l + \mathrm{i}\left[\frac{l(3l+2)\,\omega'}{4n^2\mu} + \mu\frac{(3l+1)}{8(l+1)} + O(\mu^2, \omega')\right] C_{l+1} + \sum_{m \neq l,\, l+1} O(\mu)\, C_m = 0, \tag{4.25a}$$

$$-\mathrm{i}\left[\omega'\left(1 + \frac{(2l+1)(3l+2)}{16n^2}\right) + \mu^2\left(\frac{6l^3+11l^2-4}{32l(l+1)(l+2)} + \frac{n^2}{4l(l+2)}\right) + O(\mu^3, \mu\omega')\right] C_{l+1} + \left[\left(\frac{(2l+1)\,\omega'}{8n} + O(\mu\omega')\right) J_l(a_0) - \left(\frac{(l+1)(3l+2)\,\omega'^3}{2\mu n^2} + O(\mu^2\omega')\right) J_{l+1}(a_0)\right] C_l + \sum_{m \neq l,\, l+1} O(\mu)\, C_m = 0, \tag{4.25b}$$

$$C_p + O(\mu^2, \omega')\, C_{l+1} + [O(\omega')\, J_l(a_0) + O(\mu\omega')\, J_{l+1}(a_0)]\, C_l + \sum_{m \neq l,\, l+1,\, p} O(\mu)\, C_m = 0, \quad p \neq l, l+1, \tag{4.25c}$$

where $a_0 = (\mu n/\omega')\,[1 + \tfrac{21}{16} l\omega'/n^2 + \tfrac{163}{192}\mu^2 + O(\mu\omega', \mu^3)]$.

In the case $l = 1$ (4.25) are still valid except for (4.25*c*) which at $p = 0$ has the form

$$\mathrm{i}\frac{n^2\mu^2}{2} C_0 + O(\mu^4, \omega'\mu^2)\, C_2 + [O(\omega'\,\mu^2)\, J_1(a_0) + O(\mu^3\,\omega')\, J_2(a_0)]\, C_1 + \sum_{m \neq 1,\, 2} O(\mu^3)\, C_m = 0. \tag{4.25d}$$

The difference between (4.25*d*) and (4.25*c*) relates to the fact that in the case $l = 1$ the basic displacement $\boldsymbol{\varepsilon}_{(0)}$ has a structure different from the general one (Appendix C).

In both cases $l \geqslant 2$ and $l = 1$ we easily obtain the estimates

$$C_p = O(\mu^2, \omega')\, C_{l+1} + [O(\omega')\, J_l(a_0) + O(\mu\omega')\, J_{l+1}(a_0)]\, C_1$$

at $p \neq l$, $l+1$ from (4.25*c*) or (4.25*d*). Substituting this estimate in (4.25*a*, *b*) we find that all the terms under the sign of the sum have the value of the order of the rejected terms. Thus the infinite system of algebraic equations (4.25) is reduced to a system of two equations with respect to coefficients C_l and C_{l+1}. The condition that the determinant of this system is equal to zero gives the dispersion equation

$$\left[\omega' - \mu^2\left(\frac{(6l^2+11l+6)}{32l(l+1)(l+2)} - \frac{n^2}{4l(l+2)}\right) + O(\mu^3)\right]\frac{J_1(a_0)}{J_{l+1}(a_0)} = -\frac{(3l+2)^2(l+1)\,\omega'^3}{4n^3\mu} + O(\mu^2\omega', \mu^4). \quad (4.26)$$

This dispersion equation is obtained for $l \geqslant 1$ and allows finding eigen-frequencies of a thin vortex ring which are not close to $\omega = 0$. The dispersion equation (4.26) is a transcendental equation, the roots of which are the eigen-frequencies. These roots are close to the values at which one of the left-hand-side factors becomes zero. The second factor becomes zero at the infinite number of points corresponding to zeros of the Bessel function J_l. These zeros correspond to an infinite family of eigen-oscillations with frequencies of the form

$$\omega = \tfrac{1}{2}l \pm \frac{\mu n}{a_j}[1 + O(\mu)], \quad (4.27)$$

where $J_l(a_j) = 0$, $j = 1, 2, \ldots$. These frequencies have the accumulation point $l/2$. We shall call oscillations of this family *Bessel* oscillations (similarly to the columnar vortex oscillations, §2).

At any l there exists one more eigen-oscillation corresponding to the first factor of the dispersion equation becoming zero. This oscillation has the frequency

$$\omega = \tfrac{1}{2}l + \mu^2\left(-\frac{n^2}{4l(l+2)} + \frac{(6l^2+11l+6)}{32l(l+1)(l+2)}\right) + O(\mu^3). \quad (4.28)$$

We shall call this eigen-oscillation an *isolated* one.

For the case $l = 0$ the system of equations (4.24) has the form

$$[(-\tfrac{1}{2}\mu n\omega + O(\mu^2\omega))\,J_0(a_0) + O(\mu\omega^2)\,J_1(a_0)]C_0 + O(\mu^3, \mu\omega)\,C_1 + O(\mu^3, \mu\omega)\,C_{-1} + \sum_{m \neq -1,0,1} O(\mu)\,C_m = 0, \quad (4.29a)$$

$$-\mathrm{i}\,[\omega \pm \tfrac{1}{8}\mu^2(A_n + B_n) + O(\mu^3, \mu\omega)]\,C_{\pm 1} \mp \left[O(\mu\omega)\,J_0(a_0) + \left(\frac{\omega^3}{\mu n^2} + O(\mu^2\omega)\right)J_1(a_0)\right]C_0 \mp \mathrm{i}\,[\tfrac{1}{8}\mu^2(A_n - B_n) + O(\mu^3, \mu\omega)]\,C_{\mp 1} + \sum_{m \neq -1,0,1} O(\mu)\,C_m = 0, \quad (4.29b)$$

$$C_p + [O(\mu\omega)\,J_0 + O(\mu^2\omega)\,J_1]\,C_0 + O(\mu^2)\,C_1 + O(\mu^2)\,\mathrm{C}_{-1} + \sum_{m \neq -1,0,1,p} O(\mu)\,C_m = 0, \quad p \neq -1, 0, 1, \quad (4.29c)$$

where

$$A_n = (n^2-1)\ln\frac{8}{\mu} + \frac{n^2+5}{4} - \frac{4n^2-1}{2}S_n,$$

$$B_n = n^2\ln\frac{8}{\mu} + \frac{n^2}{4} - \frac{4n^2-3}{2}S_n$$

$$S_n = \sum_{k=1}^{n}\frac{1}{2k-1}.$$

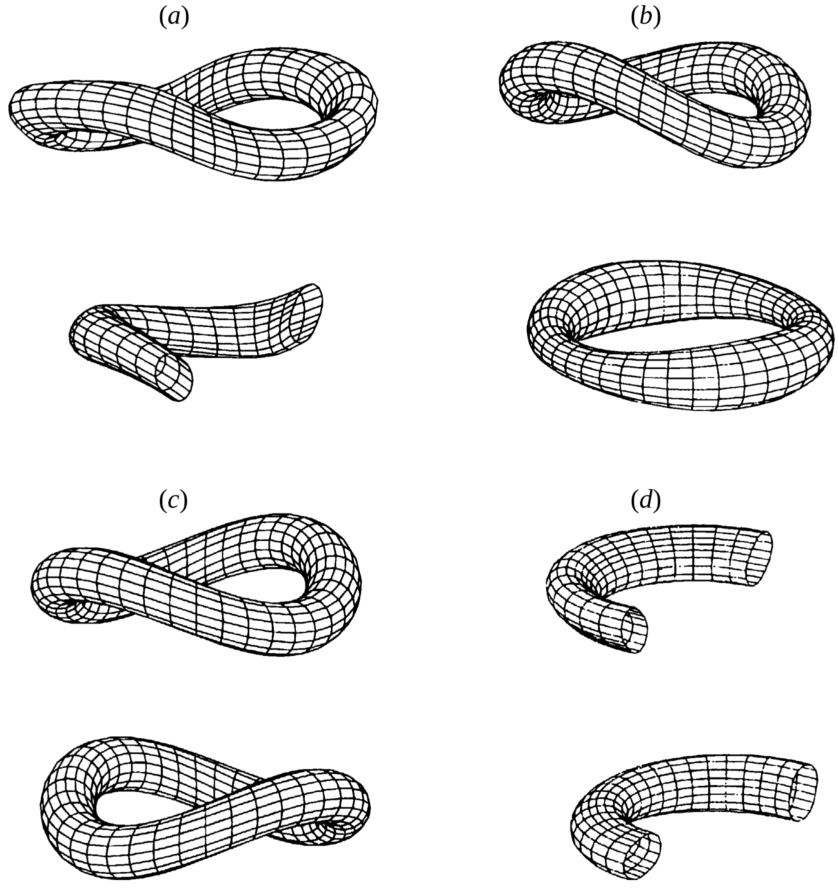

FIGURE 6. Boundary disturbance shape for three-dimensional oscillations. (*a*) Bessel mode, $l = 1$, $n = 2$: whole ring and half a ring; (*b*) bulging mode (two phases of oscillation), $l = 0$, $n = 2$, $j \geqslant 1$; (*c*) bending mode, $l = 0$, $n = 2$ (two phases of oscillation); (*d*) isolated modes, $l = 1$, $n = 1$ compared with axisymmetric mode.

In contrast to the case $l \geqslant 1$, this infinite system of algebraic equations is reduced to a system of three equations with respect to coefficients C_0, C_1 and C_{-1}. The condition that the determinant of this system is equal to zero gives the dispersion equation

$$\left[\omega^2 - \frac{\mu^4}{16} A_n B_n + O(\mu^5, \mu^3\omega)\right] \frac{J_0(a_0)}{J_1(a_0)} = O(\omega^3, \mu^6). \tag{4.30}$$

The structure of (4.26) and (4.30) is similar. Equation (4.30) has an infinite number of roots close to zeros of the Bessel function J_0. These roots correspond to the family of eigen-oscillations with the frequencies

$$\omega = \frac{\mu n}{a_j}[1 + O(\mu)], \tag{4.31}$$

where $J_0(a_j) = 0, j = 1, 2, \ldots$. These frequencies have the accumulation point $\omega = 0$. We shall call Bessel oscillations of this type *bulging* modes.

At any l there exists one more eigen-oscillation with frequency close to $\omega = 0$. It is an isolated mode (called the *bending* mode) with the frequency

$$\omega = \tfrac{1}{4}\mu^2 (A_n B_n)^{1/2} + O(\mu^3). \tag{4.32}$$

It can be seen that the structure of the oscillation spectrum of a thin vortex ring turns out to be similar to the structure of the cylindrical vortex spectrum. In fact, if the wavenumber $k = \mu n = \text{const.}$ is fixed and the ring curvature tends to zero ($\mu \to 0$), (4.27), (4.31) are exactly transformed into (2.3) at $l = m$ and (4.28), (4.32) are transformed into (2.5), (2.4) at $l = m-1$.

However, as will be shown in the next subsection, the forms of the vortex ring and vortex column oscillations are already different to the leading approximation.

4.6. *The forms of eigen-oscillations*

The accuracy of calculating the terms in the dispersion equation (4.26) and (4.30) appears to be quite sufficient for finding the whole spectrum of eigen-frequencies (up to the accumulation point). Regarding the eigen-oscillation forms, the accuracy achieved is enough only for calculation of two approximations of an isolated mode and of the leading approximation of those Bessel (bulging) modes that have frequencies furthest from the accumulation point ($\omega'/\mu = O(1)$).

Substituting each eigen-frequency in the system of algebraic equations we find the coefficient C_m of the expansion of the corresponding eigen-oscillation with respect to basic displacements $\boldsymbol{\varepsilon}_{(m)}$. Consider all the eigen-oscillations in a turn.

For Bessel oscillations at $l \geqslant 1$, $\omega'/\mu = O(1)$, the frequency of which are determined by (4.27), coefficients C_m are

$$C_l = \frac{\mathrm{i} a_j}{\mu n J_{l+1}(a_j)} + O(1), \quad C_{l+1} = -\frac{(3l+2)(l+1)}{2na_j} + O(\mu), \quad C_m = O(\mu), \quad m \neq , l+1. \tag{4.33}$$

The expressions obtained for coefficients C_m together with the expressions for the basic displacements $\boldsymbol{\varepsilon}_{(m)}$ (Appendix C) determine the form of eigen-oscillations $\boldsymbol{\varepsilon}(\sigma, \psi)$ according to (4.14). In particular, the vortex boundary disturbance $\epsilon^{\sigma}|_{\sigma=1}$ which characterizes the oscillation shape is (figure 6)

$$\epsilon^{\sigma}|_{\sigma=1} = \mathrm{e}^{\mathrm{i}l\psi} - \frac{(3l+2)(l+1)}{2na_j}\mathrm{e}^{\mathrm{i}(l+1)\psi} + O(\mu). \tag{4.34}$$

The form of Bessel oscillations of a vortex ring, to the leading approximation, appears to be a combination of two harmonics, $\exp(\mathrm{i}l\psi)$ and $\exp[\mathrm{i}(l+1)\psi]$, in contrast to the corresponding vortex column oscillations which have the form of a single harmonic $\exp(\mathrm{i}l\phi)$. The second term in (4.34) disappears in the limit $\mu \to 0$, $n \to \infty$, $k = \mu n = \text{const.}$

Thus, even a slight curvature of the vortex filament leads to dramatic changes in the characteristics of its oscillations. Moreover, velocity disturbances outside the vortex (and consequently also the sound field) are determined by the $(l+1)$th harmonic, i.e. by that part of the vortex ring disturbance which appears if the vortex filaments are curved. This is connected with the fact that the phase velocity of the boundary disturbance in the form of $\exp(\mathrm{i}l\psi - \mathrm{i}\omega t)$ is close to the mean flow velocity, i.e. this vortex boundary displacement is adhered to the mean flow and produces disturbances outside the vortex with efficiency much smaller than the $(l+1)$th harmonic with an equal amplitude (see §2).

All that has been written above about the Bessel oscillation structure at $l \geqslant 1$ concerns bulging modes (Bessel oscillations with $l = 0$). These modes have the frequency determined by (4.31). At $\omega/\mu = O(1)$ the coefficients C_m for bulging modes are

$$C_0 = \frac{\mathrm{i}a_j}{\mu n J_1(a_j)} + O(1), \quad C_1 = -\frac{1}{na_j} + O(\mu), \quad C_{-1} = \frac{1}{na_j} + O(\mu), \quad C_m = O(\mu), \quad m \neq -1, 0, 1. \tag{4.35}$$

Write only the vortex boundary disturbance form (figure 6):

$$\epsilon^{\sigma}|_{\sigma=1} = \mathrm{e}^{\mathrm{i}0\psi} - \frac{1}{na_j}(\mathrm{e}^{\mathrm{i}\psi} - \mathrm{e}^{-\mathrm{i}\psi}) + O(\mu). \tag{4.36}$$

Note an interesting feature of the standing bulging wave which is the sum of the modes with $n = \pm 1$. At different phases of such an oscillation the ring plane is inclined at different angles to the reference position. This seems to be impossible from the standpoint of the localized induction approximation (LIA) within the limits of which the ring impulse is directed along its axis. However, a more detailed analysis (Kopiev & Chernyshev 1991) taking into account the disturbance structure inside the core has shown that the flow impulse at these oscillations is well preserved in reality.

For isolated oscillations with $l \geqslant 1$, the frequency of which is determined by (4.28), coefficients C_m have the form

$$\left.\begin{aligned} &C_{l+1} = 1, \quad C_l = \mathrm{i}\frac{3l+2}{2nJ_l(a_0)} + \mathrm{i}\frac{(3l+1)\,n\mu^2}{4l(l+1)\,\omega'\,J_l(a_0)} + O(\mu), \\ &C_m = O(\mu^2), \quad m \neq l, l+1. \end{aligned}\right\} \tag{4.37}$$

For isolated oscillations with $l = 0$ (bending modes), the frequency of which is determined by (4.32), coefficients C_m have the form

$$\left.\begin{aligned} &C_1 = \tfrac{1}{2}(1-(B_n/A_n)^{1/2}) + O(\mu), \quad C_{-1} = \tfrac{1}{2}(1+(B_n/A_n)^{1/2}) + O(\mu), \\ &C_m = O(\mu), \quad m \neq -1, 1. \end{aligned}\right\} \tag{4.38}$$

As in the case of Bessel oscillations we write only the vortex boundary disturbance form (figure 6):

$$\epsilon^{\sigma}|_{\sigma=1} = \mathrm{e}^{\mathrm{i}(l+1)\psi} - \frac{\mu}{4}\mathrm{e}^{\mathrm{i}l\psi} - \frac{(2l+3)\,\mu}{4(l+1)}\mathrm{e}^{\mathrm{i}(l+2)\psi} + O(\mu^2), \quad l \geqslant 1, \tag{4.39a}$$

$$\epsilon^{\sigma}|_{\sigma=1} = \cos\psi - \mathrm{i}\,(B_n/A_n)^{1/2}\sin\psi + O(\mu), \quad l = 0. \tag{4.39b}$$

The solution for bending modes ($l = 0$) is identical to the solution already known (Widnall *et al.* 1971).

Unlike the Bessel oscillations, the isolated (bending) modes of a vortex ring, in the leading approximation, coincide with the isolated modes of a vortex column.

On the basis of the solutions obtained above in terms of the displacement field it is also easy to find the velocity disturbance field for each oscillation. In fact, the velocity field inside the vortex ($\sigma \leqslant 1$) is determined directly from (4.3a). Outside the vortex ($\sigma > 1$) the velocity v and potential Q are determined by the integral relation (4.6) in which the value of the monopole density Q, according to (4.20), has the form

$$Q = q\delta(\sigma - 1), \quad q = \sum_{m=-\infty}^{\infty} C_m\, q_m,$$

where the q_m for each basic displacement were obtained when the functions $F_{(m)}$ were calculated. The velocity potential Φ produced by monopole density of arbitrary form can be found using Appendix B (B 12).

We write the expressions for q at all eigen-oscillations. For Bessel oscillations

$$q = -\mathrm{i}\frac{(3l+2)(l+1)\,\omega'}{2n^2\mu}\,\mathrm{e}^{\mathrm{i}(l+1)\psi} + O(\mu), \quad l \geqslant 2, \tag{4.40a}$$

$$q = -\mathrm{i}\,[5\omega'/(n^2\mu) + O(\mu)]\,\mathrm{e}^{2\mathrm{i}\psi} - \mathrm{i}\,[(5/(8n^2)+2)\,\omega' + O(\mu^2)]\,\mathrm{e}^{\mathrm{i}\psi} + \mathrm{i}\,[45/(8n^2)\,\omega' + O(\mu^2)]\,\mathrm{e}^{3\mathrm{i}\psi} - \mathrm{i}\,[\mu\omega' + O(\mu^3)]\,\mathrm{e}^{\mathrm{i}0\psi} + O(\mu^2), \quad l = 1, \tag{4.40b}$$

$$q = -\mathrm{i}\left[\frac{\omega}{n^2\mu} + O(\mu)\right](\mathrm{e}^{\mathrm{i}\psi} + \mathrm{e}^{-\mathrm{i}\psi}) - \mathrm{i}\,[\omega + O(\mu^2)]\,\mathrm{e}^{\mathrm{i}0\psi} + O(\mu), \quad l = 0. \tag{4.40c}$$

For isolated oscillations

$$q = \mathrm{i}\,\mathrm{e}^{\mathrm{i}(l+1)\psi} + O(\mu), \quad l \geqslant 2, \tag{4.41a}$$

$$q = \mathrm{i}[1 + O(\mu^2)]\,\mathrm{e}^{2\mathrm{i}\psi} + \mathrm{i}\left[\frac{\mu}{8} + O(\mu^2)\right]\mathrm{e}^{\mathrm{i}\psi} - \mathrm{i}[\tfrac{9}{8}\mu + O(\mu^2)]\,\mathrm{e}^{3\mathrm{i}\psi} + O(\mu^3)\,\mathrm{e}^{\mathrm{i}0\psi} + O(\mu^2), \quad l = 1, \tag{4.41b}$$

$$q = -[1 + O(\mu)]\sin\psi - \mathrm{i}[(B_n/A_n)^{1/2} + O(\mu)]\cos\psi + O(\mu^3)\,\mathrm{e}^{\mathrm{i}0\psi} + O(\mu), \quad l = 0. \tag{4.41c}$$

Expressions (4.40) confirm the point made above about the external velocity field structure for Bessel oscillations.

Note the possibility of the appearance of the boundary displacement due to initial vorticity disturbances confined inside the core. In this case the boundary displacement (disturbing the external region) does not appear immediately but takes some time. To clarify the situation we consider two abstract oscillations $\epsilon_1 = \cos\omega_1 t$ and $\epsilon_2 = -\cos\omega_2 t$ with frequencies different from each other by an order of magnitude of μ, that is $\Delta\omega = \omega_1 - \omega_2 = O(\mu)$. At the initial moment the sum of these oscillations equals zero. However, after a not large period of time the expression $\epsilon_1 + \epsilon_2 = -2\Delta\omega t \sin\omega_1 t$ can be readily obtained, which indicates a linear increase of the oscillation amplitude, attaining $O(1)$ values for the period of time $O(1/\mu)$. A similar situation can easily arise in a vortex ring. Let the initial disturbance be located inside the core and the boundary displacement of the core be absent. The initial disturbance is expanded as a complete set of eigen-oscillations, the phase and amplitudes of which are selected in such a way that the boundary displacement is absent. Since the eigen-modes have the amplitude of a non-adherent harmonic $\exp(\mathrm{i}(l+1)\psi)$ of $O(1)$ and eigen-frequencies fit the interval $\Delta\omega = O(\mu)$, the disturbance evolution is quite similar to the case with two oscillations mentioned above. The boundary disturbance non-adherent to the main flow arises at the time $O(1/\mu)$. This linear increase of the sound-generating oscillation amplitude is not produced by the usual instability (eigen-oscillation with complex frequency) and is connected only with the structure of the initial disturbance which can readily appear at the moment of the vortex sheet convolution into a vortex core. This phenomenon is rather similar to algebraic instability.

Special attention must be given to the possibility of degeneration of the isolated (bending) and Bessel (bulging) modes (the coincidence of zeros of the first and the second factors on the left-hand side of the dispersion equation (4.26) or (4.30) at certain parameters of the steady flow). This situation is possible at $\omega' = O(\mu^2)$. According to the sign of the right-hand side of the dispersion equation, the appearance of instability

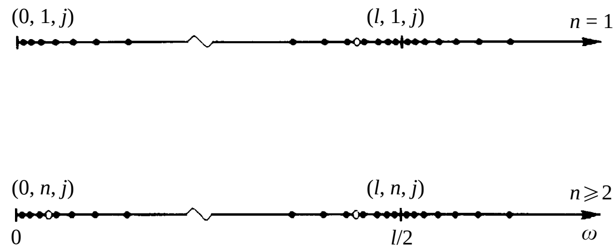

FIGURE 7. Spectrum of three-dimensional vortex ring oscillations (the numbers l, n and j are indicated in brackets). ●, Bessel oscillations (including bulging mode); ○, isolated oscillations (including bending mode).

or beating is possible (Landau & Lifshitz 1979, §6.4). In the case the appearance of beating between the bending and bulging modes a very interesting phenomenon can be expected in the vortex ring. Let the degeneration of a bending mode and one of the bulging modes of the vortex ring take place and let the initial state of the ring be in the form of flexure. Then after a time all the energy of this bending disturbance will transfer into a bulging disturbance with a large value of j due to the beating effect. The characteristic scale of the bending disturbance has the order of the vortex radius R. At the same time the characteristic scale of the bulging disturbance is determined by this disturbance structure in the coordinate ρ in the core section (i.e. it is determined by j). Therefore the characteristic scale of a bulging disturbance will be the cross-section size a divided by j, i.e. $\mu^2 R$. In the presence of even a small viscosity such energy transfer between the bending and bulging disturbances will go only in one direction since the bulging disturbances will rapidly dissipate because of their small-scale structure.

4.6.1. *Classification of three-dimensional eigen-oscillations*

Long-wave three-dimensional eigen-oscillations of the isochronous vortex ring can be characterized by three integers: the frequency number l, azimuthal number n and radial number j. In fact, the vortex ring oscillations have frequencies located in the neighbourhood of $\omega = l/2$ and they have a definite number n of the azimuthal harmonic. Besides, Bessel and bulging oscillations differ by the radial number $j \geqslant 1$ which characterizes the disturbance form in the vortex core section. In accordance with this, we characterize Bessel (bulging) oscillations by three numbers (l, n, j), and each isolated (bending) oscillation by the numbers $(l, n, 0)$ (the radial number for the isolated modes is taken equal to zero). Without losing generality we confine ourselves to consideration of the positive frequencies ($l \geqslant 0$) and non-negative azimuthal numbers ($n \geqslant 1$). As it is shown above, these solutions can be readily generalized to the other parameter ranges.

Thus Bessel and bulging oscillations may have the frequency number $l \geqslant 0$ and the azimuthal number $n \geqslant 1$, the isolated oscillations (except the bending mode) have the frequency number $l \geqslant 1$ and the azimuthal number $n \geqslant 1$, while the bending mode exists only at $l = 0$ and $n \geqslant 2$ (figure 7).

4.7. *Effect of the shape of the vorticity profile of the steady vortex ring on the properties of its oscillations*

The two-dimensional (axisymmetrical) eigen-oscillations of a vortex ring with uniform vorticity (§§3.2, 3.3) and the three-dimensional eigen-oscillations of an isochronous vortex ring (§4.6) have been obtained. The question of the applicability of these solutions to any similar steady vortex rings now arises.

The complete solution of the problem of the eigen-oscillations of a vortex ring with arbitrary vorticity profiles is one of extreme complexity since the spectrum of these disturbances in a general case includes both discrete and continuous components. Nevertheless, the solutions obtained in the present work for two of the simplest vorticity profile shapes prove to be quite sufficient for understanding the evolution during rather a long period of time of arbitrary disturbances of any vortex rings, the stationary flow of which is similar to that considered above.

In fact, the exact expressions for the eigen-oscillations are not always necessary for a description of the initial disturbance evolution of any oscillatory system. Such a description can often be achieved on the basis of a consideration of a similar system (if it is more simple), the eigen-oscillations of which are considered as slightly interacting. A similar situation occurs in the well-known problem related to weakly connected oscillators. It can be solved both in the language of exact eigen-oscillations and in the language of individual oscillator eigen-oscillations weakly interacting with each other.

Therefore, the solutions obtained for the uniform vortex ring and the isochronous one can be used for describing the disturbance evolution of any similar stationary vortex ring if the solutions obtained are considered to be weakly interacting. A quantitative description of this interaction with the use of perturbation methods is outside the scope of this paper. However, the idea of the disturbance evolution of an arbitrary vortex ring, as the result of interaction of the vortex ring modes obtained above, permits the evaluation of the time during which this interaction can not be taken into account in the initial value problem.

Consider for example any eigen-oscillation for an isochronous vortex ring. Substitute this solution into the system of equations (4.4) in which the stationary flow differs by an amount of $O(\mu^2)$ from the isochronous one. This equation can be readily reduced to the evolutionary equation with an isochronous profile and with the right-hand side of $O(\mu^2)$. This means that such an initial disturbance keeps its form during a large time interval and is distorted after a time of $O(\mu^{-2})$ due to appearance of the secular terms. This can be extended to an arbitrary initial disturbance since the system of axisymmetrical and three-dimensional eigen-oscillations is complete.

Thus, the evolution of an arbitrary initial disturbance of any stationary ring differing from uniform (isochronous) by an amount of $O(\mu^2)$ can be described for a length of time of $O(\mu^{-2})$ with the help of the solutions found in §§3.2, 3.3 and 4.6.

5. Sound radiation by vortex ring oscillations

It is well known that in a compressible fluid the unsteady movement of vortices is accompanied by sound wave radiation. If the sound wavelength considerably exceeds the vortex size, the sound field is determined by the vortex dynamics which is obtained in the incompressible fluid approximation. The simplest expression connecting the sound field with incompressible vortex dynamics was obtained by Möhring (1978) and is as follows:

$$p(\boldsymbol{r}, t) = \frac{M^2}{4\pi r} N_i N_j \frac{\mathrm{d}^3}{\mathrm{d}t^3} D_{ij}(t - Mr), \quad N_i = \frac{r_i}{r}, \tag{5.1}$$

$$D_{ij}(t) = \frac{1}{3}\int [\boldsymbol{r} \times \boldsymbol{\Omega}]_i \, r_j \, \mathrm{d}\boldsymbol{r}, \tag{5.2}$$

where $M = \Omega_0 a / c_0$, c_0 is the sound speed. Dimensionless variables are used. The length and time scales are determined as in §3.1. Also, the undisturbed density is set to unity.

In this section the sound fields generated by the vortex ring oscillations are calculated. For the calculation we use the solutions obtained in §§3 and 4. It has been pointed out in §4.7 that one can consider these solutions as a typical complete set of eigen-oscillations for any vortex ring with vorticity slightly different from uniform or isochronous ones.

The main problem arising in the calculation of the sound field produced by vortex ring oscillations consists in the following. The vortex ring eigen-oscillations in incompressible fluid are found as an expansion in harmonics $\exp(\mathrm{i}m\psi)$. The amplitude of each harmonic is known only with some accuracy in the parameter μ. It appears that harmonics with a different order in μ can make the same contribution to the sound field, depending on the harmonic number m. Therefore a careful analysis of the contribution of all the eigen-oscillation components to the sound field is required so that it can be stated with certainty that all the necessary terms have been taken into account.

In this section the integral (5.2) is transformed into the surface one. Integration over θ for oscillations of the type $\exp(\mathrm{i}n\theta)$ can be done in a general way. As a result, the volume integral (5.2) determining a quadrupole moment is reduced to only the integral in the angular variable ψ. Such a representation permits the evaluation of the contribution of different terms of the eigen-oscillations to the sound field in a rather simple manner.

5.1. *Transformations of the formula for the sound field*

Since contractions of both tensor $r_i[\boldsymbol{r}\times\boldsymbol{\Omega}]_j$ and $r_j[\boldsymbol{r}\times\boldsymbol{\Omega}]_i$ with the symmetrical tensor $N_i N_j$ give an identical result, then in calculating the sound field one can use the following expression instead of (5.2):

$$D_{ij} = \frac{1}{6}\int [\boldsymbol{r}\times\boldsymbol{\Omega}]_i\, r_j\, \mathrm{d}\boldsymbol{r} + \frac{1}{6}\int [\boldsymbol{r}\times\boldsymbol{\Omega}]_j\, r_i\, \mathrm{d}\boldsymbol{r}. \tag{5.3}$$

Substitute (4.2) into (5.3). After simple transforms we get

$$D_{ij} = -\frac{\delta_{ij}}{3}\int r_k[\boldsymbol{\varepsilon}\times\boldsymbol{\Omega}_0]_k\, \mathrm{d}\boldsymbol{r} + \frac{1}{2}\int (r_i[\boldsymbol{\varepsilon}\times\boldsymbol{\Omega}_0]_j + r_j[\boldsymbol{\varepsilon}\times\boldsymbol{\Omega}_0]_i)\, \mathrm{d}\boldsymbol{r}. \tag{5.4}$$

It is easy to check that the tensor identity holds:

$$r_i r_j Q = r_i[\boldsymbol{\varepsilon}\times\boldsymbol{\Omega}_0]_j + r_j[\boldsymbol{\varepsilon}\times\boldsymbol{\Omega}_0]_i - 2\delta_{ij}G - \frac{\partial}{\partial x_k}[r_i r_j(\boldsymbol{\varepsilon}\times\boldsymbol{\Omega}_0 + \boldsymbol{\nabla}G)_k] + \frac{\partial}{\partial x_j}(r_i G) + \frac{\partial}{\partial x_i}(r_j G), \tag{5.5}$$

where the functions Q and G are determined in (4.7). Substitute (5.5) into (5.4) and transform the volume integrals from the terms with derivatives into surface ones. Since outside the vortex the functions $\boldsymbol{\Omega}_0$ and G are identically equal to zero, the surface integrals are also equal to zero. Thus (5.4) can be written

$$D_{ij} = \frac{1}{2}\int r_i r_j Q\, \mathrm{d}\boldsymbol{r} - \frac{\delta_{ij}}{6}\int r^2 Q\, \mathrm{d}\boldsymbol{r}. \tag{5.6}$$

An expression for a quadrupole moment similar to (5.6) without the second term was used by Kopiev & Chernyshev (1993). This was possible since the second term became zero for the oscillations considered there. In a general case the correct equation (5.6) must be used.

On choosing G according to (4.21) the integration over the σ-variable becomes trivial, since Q is a surface density of the form $Q = q(\psi)\,\delta(\sigma-1)$. The q for all types of three-dimensional vortex ring eigen-oscillations were obtained in §4.6.

Write down the radius vector $\boldsymbol{r}$ (figure 3) in the form

$$\boldsymbol{r} = \rho\sin\phi\,\boldsymbol{e}_z + (\mu^{-1} - \rho\cos\phi)\,\boldsymbol{e}_\xi,$$

where $(\boldsymbol{e}_\xi)_x = \cos\theta$, $(\boldsymbol{e}_\xi)_y = \sin\theta$. Substitute the components of radius vector $\boldsymbol{r}$ in (5.6) and integrate over the angle θ. Then, making use of the dependence on $\exp(\mathrm{i}n\theta)$ of Q, we obtain that the quadrupole moment D_{ij} differs from zero only for three azimuthal numbers $n = 0, 1, 2$. Contracting the tensors D_{ij} and $N_i N_j$ for these values of n we get the radiation directivity:

$$N_i N_j D_{ij} = \pi\mu^{-3}(\tfrac{1}{2}(I_1+I_3) - \tfrac{1}{3}I_4 + \tfrac{1}{2}(I_3 - I_1)\cos 2\chi), \quad n = 0, \tag{5.7a}$$

$$N_i N_j D_{ij} = \pi\mu^{-3} I_2\,\mathrm{e}^{\mathrm{i}\theta}\sin 2\chi, \quad n = 1, \tag{5.7b}$$

$$N_i N_j D_{ij} = \pi\mu^{-3}\tfrac{1}{2}I_1\,\mathrm{e}^{2\mathrm{i}\theta}\sin^2\chi, \quad n = 2. \tag{5.7c}$$

The constants I_1, I_2, I_3, I_4 are the integrals over the ψ-variable, the values of which depend on the particular form of $q(\psi)$:

$$I_1 = \int_0^{2\pi} [1 - 2\mu\cos\psi + \tfrac{1}{4}\mu^2(1 + 3\cos 2\psi) + O(\mu^3)]\, q(\psi)\,\mathrm{d}\psi, \tag{5.8a}$$

$$I_2 = \int_0^{2\pi} \mu[\sin\psi - \tfrac{5}{8}\mu\sin 2\psi + O(\mu^2)]\, q(\psi)\,\mathrm{d}\psi, \tag{5.8b}$$

$$I_3 = \int_0^{2\pi} \mu^2[1 - \cos 2\psi + O(\mu)]\, q(\psi)\,\mathrm{d}\psi, \tag{5.8c}$$

$$I_4 = \int_0^{2\pi} [1 - 2\mu\cos\psi + \tfrac{1}{4}\mu^2(3 + \cos 2\psi) + O(\mu^3)]\, q(\psi)\,\mathrm{d}\psi. \tag{5.8d}$$

The expressions (5.8) readily permit the evaluation of the contribution of different harmonics $\exp(\mathrm{i}p\psi)$ entering the function $q(\psi)$ to the sound field. This permits one to find the necessary accuracy with which various harmonics must be calculated. For example, the following harmonics give identical contributions to the integrals (5.8): harmonic $\exp(2\mathrm{i}\psi)$ of $O(1)$, harmonic $\exp(\mathrm{i}\psi)$ of $O(\mu)$ and harmonic $\exp(0\mathrm{i}\psi)$ of $O(\mu^2)$. This was why the calculation of different harmonics in $q(\psi)$ was done with different accuracy in §4.6.

5.2. *Calculation of the sound field produced by vortex ring eigen-oscillations*

The many eigen-oscillations of the vortex ring can be divided into three groups according to the value of the eigen-frequencies: they are bending modes ($\omega = O(\mu^2\ln\mu)$), bulging modes ($\omega = O(\mu)$) and several families of oscillations with frequency $\omega = O(1)$. For each group we single out those oscillations the sound radiation of which has the largest quadrupole moment according to the order of magnitude.

As it was shown in the previous section, the quadrupole moment is different from zero only for those oscillations which have azimuthal numbers $n = 0, 1, 2$. Besides, among the oscillations with the frequency $\omega = O(1)$ the most efficient ones are those which have the frequency number $l = 1$. Indeed, with increase in l the number of main harmonics which determine the oscillation form and function $q(\psi)$, in the leading approximation, also increases. In turn, the larger the number of the harmonic entering the function $q(\psi)$, the smaller is the contribution of this harmonic to integrals (5.8) and, hence, to the quadrupole moments (5.7). In particular, direct substitution of (4.40*a*) into (5.8) in the case of $l \geqslant 2$ gives a value of integrals (5.8) not larger than $O(\mu^4)$ and a substitution of (4.40*b*) into (5.8) in the case of $l = 1$ gives the value $O(\mu^3)$.

Thus, only eigen-oscillations with $n = 0$, 1, 2 and $l = 0$, 1 are worthy of attention. For three-dimensional oscillations ($n = 1, 2$) we need to consider the modes $(1, 1, j)$, $(1, 2, j)$, (0, 1, j) and $(0, 2, j)$, where $j \geqslant 0$ (§4.6). For axisymmetrical modes ($n = 0$) we need to consider a complete set of modes with $l = 1$, including the continuous spectrum modes.

5.2.1. *Bessel oscillations*

Consider Bessel oscillations of the type $(1, 1, j)$ and $(1, 2, j)$, where $j \geqslant 1$. To calculate the sound field the values of q determined by (4.40*b*) are substituted in (5.8*b*) for $n = 1$ and (5.8*a*) for $n = 2$. Then for the oscillation $(1, 1, j)$ we obtain

$$p = \pm \frac{\mathrm{i}\pi M^2}{2^6 a_j \mu} \frac{\mathrm{e}^{\mathrm{i}Mr/2}}{r} \mathrm{e}^{\mathrm{i}\theta} \sin 2\chi. \tag{5.9}$$

For the oscillations $(1, 2, j)$ we obtain

$$p = \pm \frac{11\pi M^2}{2^8 a_j \mu} \frac{\mathrm{e}^{\mathrm{i}Mr/2}}{r} \mathrm{e}^{2\mathrm{i}\theta} \sin^2 \chi. \tag{5.10}$$

The signs + and − in the formulas for the sound field are associated with the fact that the frequencies of Bessel oscillations (4.27) are both to the left and to the right of the value $\omega = l/2$.

5.2.2. *Bulging oscillations*

Consider bulging oscillations of the type $(0, 1, j)$ and $(0, 2, j)$. To calculate the sound field the values of q determined by (4.40*c*) are substituted in (5.8*b*) for $n = 1$ and (5.8*a*) for $n = 2$. Substituting (4.40*c*) into (5.8*b*) we get the integral I_2 at $O(\mu^2)$, since the $O(\mu)$ terms give exactly zero at integration. Hence for the oscillation $(0, 1, j)$ the sound field can be evaluated as

$$p = C \frac{\mathrm{e}^{\mathrm{i}\mu Mr/a_j}}{r} \mathrm{e}^{\mathrm{i}\theta} \sin 2\chi, \tag{5.11}$$

where $C = O(\mu^2 M^2)$.

For the oscillations $(0, 2, j)$ we obtain

$$p = \frac{2\pi M^2 \mu}{a_j^4} \frac{\mathrm{e}^{\mathrm{i}2\mu Mr/a_j}}{r} \mathrm{e}^{2\mathrm{i}\theta} \sin^2 \chi. \tag{5.12}$$

Note that the sound field amplitude depends on the radial number j to a large degree ($p \sim 1/a_j^4$). Therefore a considerable contribution to the sound field can be made only by the oscillations with not large radial numbers j.

5.2.3. *Isolated modes*

Consider isolated oscillations of the type (1, 1, 0) and (1, 2, 0). To calculate the sound field the values of q determined by (4.41*b*) are substituted in (5.8*b*) for $n = 1$ and (5.8*a*) for $n = 2$. Then for the oscillation (1, 1, 0) we obtain

$$p = \frac{\mathrm{i}\pi M^2}{2^6 \mu} \frac{\mathrm{e}^{\mathrm{i}Mr/2}}{r} \mathrm{e}^{\mathrm{i}\theta} \sin 2\chi. \tag{5.13}$$

For the oscillation $(1,2,0)$ we obtain

$$p = -\frac{\pi M^2}{2^7 \mu}\frac{e^{iMr/2}}{r} e^{2i\theta} \sin^2\chi. \tag{5.14}$$

5.2.4. *Bending oscillations*

Consider bending oscillation of the type $(0,2,0)$. To calculate the sound field the values of q determined by (4.41 *c*) are substituted in (5.8 *a*). Then we obtain

$$p = -\frac{\pi M^2 \mu^4 B_n^2 A_n}{2^8}\frac{\exp(i(A_n B_n)^{1/2} M\mu^2 r/4)}{r} e^{2i\theta}\sin^2\chi, \tag{5.15}$$

where A_n and B_n are determined from (4.29).

5.2.5. *Axisymmetric oscillations*

Consider isolated mode (3.8) and an infinite set of modes of the continuous spectrum (3.16) located inside the interval of $O(\mu^2)$ near $\omega = l/2$ $(l = 1)$. To obtain the values of q for these oscillations the standard procedure similar to the case of three-dimensional oscillations (§4.4) is used. For all these modes the value of q appears to be the same and is

$$q = i[1 + O(\mu^2)]\, e^{2i\psi} + i\left[\frac{\mu}{8} + O(\mu^2)\right] e^{i\psi} - i\left[\frac{9}{8}\mu + O(\mu^2)\right] e^{3i\psi} + O(\mu^2). \tag{5.16}$$

Comparison of (5.16) and (4.41 *b*) shows that the value of q for axisymmetric oscillations of the uniform vortex ring coincides with the value of q for three-dimensional isolated oscillations of an isochronous vortex ring, except for the harmonic $\exp(0i\psi)$ which is absent in (5.16).

To calculate the sound field the values of q determined by (5.16) are substituted in (5.8). Then for all the axisymmetric modes we obtain

$$p = -\frac{\pi M^2}{2^6 \mu}\frac{e^{iMr/2}}{r}(3\cos^2\chi - 1). \tag{5.17}$$

5.2.6. *Comparison of vortex ring eigen-oscillations according to their sound radiation efficiency*

The frequency ω enters (5.1) for the sound field in the third power. Therefore the modes which have frequencies of different orders in the parameter μ have substantially different radiation efficiency.

Thus, Bessel, isolated and axisymmetric modes at $l = 1$ relate to fast oscillations. For these oscillations the sound pressure p is of $O(\mu^{-1})$. Bulging and bending modes relate to slow oscillations. Therefore the sound radiation efficiency of these modes is significantly lower. In their turn, the frequencies of bulging and bending oscillations differ by an order of magnitude. According to this, p for bulging oscillations is of $O(\mu)$ and for bending oscillations it is of $O(\mu^4 \ln^3 \mu)$.

Thus, the modes which provide the most efficient sound radiation are the infinite number of modes with the identical frequency number $l = 1$. They are two isolated modes, and many Bessel modes and axisymmetric modes. These modes are of similar frequencies. The frequencies of axisymmetric modes are in the interval $\Delta\omega/\omega = O(\mu^2)$, the frequencies of Bessel modes are in the interval $\Delta\omega/\omega = O(\mu)$.

The sound radiation by bulging and bending modes appears to be of low efficiency. Nevertheless, sound radiation of these modes can be of interest in those cases when low-frequency radiation is important.

6. Discussion

Consider the possibility of applying these results to explain an acoustic experiment with a turbulent vortex ring. First, note that a turbulent vortex ring is substantially different from the ideal analogue which the theory deals with. The ideal vortex ring (figure 3) is actually a vortex torus, surrounded by an ellipsoidal region (the so-called vortex envelope) in which the streamlines are closed but the fluid is not vortical (at certain parameters of the ring the vortex envelope degenerates (Batchelor 1970, figure 7.2.4)). The toroidal vortex core together with the envelope move without variation of form and geometric size. The flow dynamics is determined by the vortex core which is an oscillating system with an infinite number of degrees of freedom (§§3, 4). If eigen-oscillations are excited, the forward movement of the ring is accompanied by sound radiation which is calculated in §5.

Experiments show that the real ring movement at high Reynolds number is significantly more complex (see, for example, Shariff & Leonard 1992). First, the flow appears to be turbulent and is accompanied by an intensive wake (figure 8). Secondly, the movement is accompanied by a slow variation of geometric vortex parameters, vorticity in the core, forward velocity, etc. However, in the work of Vladimirov & Tarasov (1979) and in the recent work of Johari (1995) it has been shown that the structure of turbulence in the turbulent vortex ring is substantially non-uniform. Thus, only the flow in the vortex envelope and in the wake proves to be turbulent while in the vortex core the turbulence is suppressed.

Since the laminar vortex core seems to be dominant in the overall vortex dynamics, the turbulence suppression in the core allows one to expect that, in the leading approximation, the laminar theory presented in this paper will be true.

At present it has been found experimentally that a solitary turbulent vortex ring radiates sound and this radiation level is sufficient for its reliable measurement (Zaitsev *et al.* 1990). The vortex rings investigated in that paper were generated in an anechoic chamber by means of a piston-driven vortex generator with a nozzle diameter $d = 4$ cm and an initial jet ejection velocity $V_0 \approx 30$ m s^{-1} (corresponding to Reynolds number $Re = V_0 d/\nu = 6.8 \times 10^4$). The ring noise was determined from the averaged spectrum in a series of 12 selected time samples of length 31.2 ms starting after 220 ms from the initiation of the ring (this corresponds to the part of the path at a distance from 200 to 230 cm from the nozzle orifice), manifesting itself as strong peaking of the spectrum in a narrow frequency band ($\Delta\omega \approx 300$ Hz) with the maximum near the frequency $\omega_0 \approx 1200$ Hz (figure 9). During the time samples of acoustic measurements the vortex ring has the following measured parameters $R_0 \approx 3.5$ cm, $\mu \approx 0.12$, $V_0 \approx 10$ m s^{-1}. The peak frequency drift connected with a slow variation of the properties of the vortex core for the time sample is $\Delta\omega \approx 60$ Hz.

The detailed analysis of three independent experiments (Zaitsev & Kopiev 1991) – measurements of the sound field, measurements of the vortex velocity and visualization – has led to a paradoxical situation. On the one hand, the peak frequency exactly agrees with the frequency of the isolated mode with $l = 1$. On the other hand, the specific characteristics of the signal are found to be typical of a narrow-band stochastic process rather than of a harmonic component (as it could be for radiation by a separate mode).

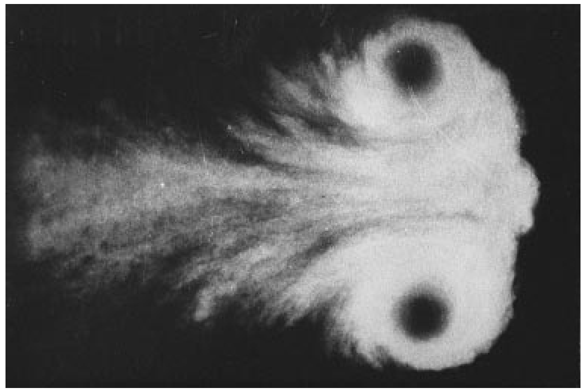

FIGURE 8. Cross-section of a turbulent smoke ring, side view (Kopiev *et al.* 1996).

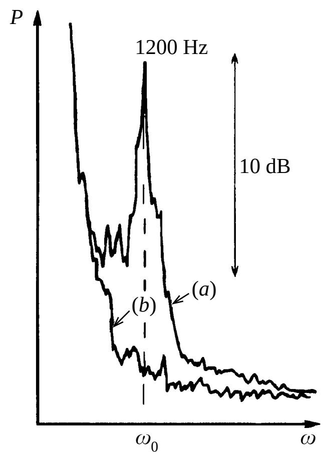

FIGURE 9. Averaged sound pressure spectrum: (*a*) with a vortex ring present; (*b*) without a vortex ring.

Within the limits of the theory derived one can readily explain these experimental facts: peak existence, uniqueness of peak, peak width and narrow band stochastic feature of the signal. The modes with $l = 1$ do indeed radiate sound most efficiently. On the other hand, $l = 1$ corresponds not to one but to a number of modes: two isolated modes with frequencies $\omega = \frac{1}{2} + O(\mu^2)$, two families of Bessel modes with frequencies $\omega = \frac{1}{2} + O(\mu)$ and axisymmetric modes with frequencies $\omega = \frac{1}{2} + O(\mu^2)$. These modes are in the interval $\Delta\omega/\omega = (-4\mu/a_1, 4\mu/a_1)$ relative to the central point $\omega_0 = \frac{1}{2}$. In dimensional variables for $\mu = 0.12$, $a_1 = 3.83$ and $\omega_0 \approx 1200$ Hz we obtain $\Delta\omega =$ 300 Hz, i.e. the peak radiation width agrees with the theoretical results not only qualitatively but quantitatively as well. Obviously the mixed sound radiation of these modes will be similar to a narrow-band stochastic process.

Finally, some comments should be made concerning possible mechanisms of excitation of the sound-generating oscillations in the vortex core. One of such mechanism can be associated with oscillation instabilities. At present three types of vortex ring instability are known: bending mode instability (Widnall & Tsai 1977; Saffman 1978), bulging mode instability (Kopiev & Chernyshev 1995), and acoustic instability (Kopiev & Leontiev 1987). In the first two cases the unstable modes appear to be non-sound-generating ones and therefore to explain the phenomenon of excitation of sound-generating oscillations it would be necessary in this case to involve the nonlinear effects. Acoustic instability has a small increment of $O(M^5)$. Another possible source of sound-generating oscillation excitation is initial disturbances. Note the possibility of sound-generating disturbance excitation not immediately after the ring generation but after some period of time due to the evolution of the initial disturbances confined inside the vortex core (§4.6).

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Appendix A

The coordinate system σ, ψ, s satisfying (3.3) for both types of steady flow considered in the present work (uniform and isochronous) with accuracy up to the terms of $O(\mu^2)$ is determined by (3.4) and has a metrical tensor of the following form:

$$g_{ij} = G_{ij} + \sum_k g_{ij}^{(k)} \mathrm{e}^{\mathrm{i}k\psi}, \quad G_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sigma^2 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$g_{ij}^{(1)} = \mu \begin{pmatrix} \frac{5}{4}\sigma & \mathrm{i}\frac{3}{4}\sigma^2 & 0 \\ \mathrm{i}\frac{3}{4}\sigma^2 & -\frac{1}{4}\sigma^3 & 0 \\ 0 & 0 & -\sigma \end{pmatrix} + O(\mu^3), \quad g_{ij}^{(0)} = \mu^2 \begin{pmatrix} \frac{131}{32}\sigma^2 & 0 & 0 \\ 0 & \frac{21}{32}\sigma^4 & 0 \\ 0 & 0 & -\sigma^2 \end{pmatrix} + O(\mu^3),$$

$$g_{ij}^{(2)} = \mu^2 \begin{pmatrix} -\frac{3}{8}\ln(8/\mu) + \frac{15}{64} + \frac{9}{16}\sigma^2 & \mathrm{i}(-\frac{3}{8}\ln(8/\mu) + \frac{15}{64})\sigma + \mathrm{i}\frac{3}{8}\sigma^3 & 0 \\ \mathrm{i}(-\frac{3}{8}\ln(8/\mu) + \frac{15}{64})\sigma + \mathrm{i}\frac{3}{8}\sigma^3 & (\frac{3}{8}\ln(8/\mu) - \frac{15}{64})\sigma^2 - \frac{3}{16}\sigma^4 & 0 \\ 0 & 0 & \frac{3}{8}\sigma^2 \end{pmatrix} + O(\mu^3),$$

$$g_{ij}^{(k)} \leqslant O(\mu^3), \quad k \geqslant 3, \quad g_{ij}^{(-k)} = (g_{ij}^{(k)})^*.$$

Appendix B

First, find the velocity field $\boldsymbol{v}$ generated by sources with density

$$Q(\sigma, \psi, \theta) = \delta(\sigma - 1)\, \mathrm{e}^{\mathrm{i}m\psi + \mathrm{i}n\theta}, \tag{B 1}$$

where $\delta(x)$ is Dirac's delta-function. The velocity field in the region $\sigma > 1$ is expressed through the source density Q as

$$\boldsymbol{v} = \boldsymbol{\nabla}\Phi, \quad \Phi = -\frac{1}{4\pi}\int \frac{Q(\boldsymbol{r}')\,\mathrm{d}\boldsymbol{r}'}{|\boldsymbol{r} - \boldsymbol{r}'|}. \tag{B 2}$$

The magnitude of the radius vector from the point of integration $\boldsymbol{r}'$ to the point of observation $\boldsymbol{r}$ is written

$$|\boldsymbol{r} - \boldsymbol{r}'|^2 = R_c^2(1 - \mu_c d)(1 - \mu_c d')[2(1 - \cos(\theta - \theta')) + \gamma^2], \tag{B 3}$$

$$\gamma^2 = \mu_c^2 \frac{(h - h')^2 + (d - d')^2}{(1 - \mu_c d)(1 - \mu_c d')}, \tag{B 4}$$

where d, h are the Cartesian coordinates in the ring cross-section plane, $d = \rho_c \cos\phi_c$, $h = \rho_c \sin\phi_c$, $\mu_c = \mu R/R_c$, R is the distance from the ring axis to the stagnation point in the ring cross-section, R_c is the distance from the ring axis to the centre of the vortex core, $R/R_c = 1+(5/8)\mu^2+O(\mu^3)$. The polar coordinates ρ_c and ϕ_c are associated with the ring cross-section centre. The use of these coordinates seems to be convenient for calculating the integral (B 2). Substituting (B 3) into (B 2) we get for the density of (B 1)

$$\Phi = -\frac{R\,\mathrm{e}^{\mathrm{i}n\theta}}{2\pi R_c(1-\mu_c\rho_c\cos\phi_c)^{1/2}}\int_0^1\int_0^{2\pi}\frac{\delta(\sigma'-1)\,\mathrm{e}^{\mathrm{i}m\psi'}\,P(\sigma,\psi,\sigma',\psi')\,\sigma'\,\mathrm{d}\sigma'\,\mathrm{d}\psi'}{(1-\mu_c\rho_c'\cos\phi_c')^{1/2}}, \quad \text{(B 5)}$$

where

$$P = \int_0^{\pi}\frac{\cos n\alpha\,\mathrm{d}\alpha}{[2(1-\cos\alpha)+\gamma^2]^{1/2}}. \quad \text{(B 6)}$$

The integral P can be expressed for any n through the sum of elliptical functions (Gradshteyn & Ryzhik 1980). To obtain an asymptotic expansion of the integral P in the parameter μ it is possible to use this sum. Each term of the expansion of the integral P in μ in such a representation is a series of corresponding terms in the expansion of each elliptic function. However, the summing of these series is a very cumbersome procedure. For the first two terms in μ a more simple way appears to exist. To this end, we divide the integration region into two parts: $0 \leqslant \alpha < c$ and $c \leqslant \alpha < \pi$, where c is an arbitrary constant with the value $\mu \ll c \ll 1$. In accordance with this separation the integral (B 6) is presented as a sum of integrals over each of these regions, i.e. $P = P_1+P_2$. The integrand in P_1 is expanded in $\alpha \ll 1$. On integrating we get

$$P_1 = \int_0^{c}\frac{\cos n\alpha\,\mathrm{d}\alpha}{[2(1-\cos\alpha)+\gamma^2]^{1/2}} = \ln\frac{2c}{\gamma}+\left(\frac{1}{48}-\frac{n^2}{4}\right)c^2$$
$$+\gamma^2\left[\frac{4n^2-1}{16}\ln\frac{2c}{\gamma}+\left(\frac{5}{96}-\frac{n^2}{8}\right)+\frac{1}{4c^2}-\frac{3}{32}\frac{\gamma^2}{c^4}\right]+O(c^4)+O\left(\frac{\gamma^4}{c^2}\right)+O\left(\frac{\gamma^6}{c^6}\right). \quad \text{(B 7)}$$

The integrand in P_2 is expanded in $\gamma \ll 1$. On integrating we get

$$P_2 = \int_c^{\pi}\frac{\cos n\alpha\,\mathrm{d}\alpha}{[2(1-\cos\alpha)+\gamma^2]^{1/2}} = -2S_n-\ln\frac{c}{4}+\left(\frac{n^2}{4}-\frac{1}{48}\right)c^2$$
$$+\gamma^2\left[-\frac{1}{4c^2}+\left(\frac{3}{8}n^2+\frac{1}{96}\right)-\frac{4n^2-1}{8}S_n-\frac{4n^2-1}{16}\ln\frac{c}{4}+\frac{3}{32}\frac{\gamma^2}{c^4}\right]+O(c^4)+O\left(\frac{\gamma^4}{c^2}\right)+O\left(\frac{\gamma^6}{c^6}\right), \quad \text{(B 8)}$$

$$S_n = \sum_{k=1}^{n}\frac{1}{2k-1}. \quad \text{(B 9)}$$

Let $c = O(\mu^p)$, $1/2 < p < 2/3$. Then adding (B 7) and (B 8) we get

$$P = -2S_n+\ln\frac{8}{\gamma}+\gamma^2\left[\frac{4n^2-1}{16}\ln\frac{8}{\gamma}+\frac{4n^2+1}{16}-\frac{4n^2-1}{8}S_n\right]+O(\mu^q), \quad \text{(B 10)}$$

where $q > 2$.

Substituting (B 10) into (B 5) and integrating over σ' and ψ' we obtain the velocity potential in the region $\sigma > 1$:

$$\Phi = -\ln\frac{8}{\mu\rho_c}+2S_n+\left(-\ln\frac{8}{\mu\rho_c}+2S_n+1\right)\frac{\mu\rho_c}{2}\cos\phi_c-\frac{3\mu}{8\rho_c}\cos\phi_c+O(\mu^2), \quad m = 0, \quad \text{(B 11)}$$

$$\Phi = -\frac{1}{2\rho_c}\mathrm{e}^{\mathrm{i}\phi_c} + \frac{\mu}{4}\left(-\ln\frac{8}{\mu\rho_c} + 2S_n + \frac{1}{2}\right) - \frac{\mu}{8}\left(1 + \frac{1}{\rho_c^2}\right)\mathrm{e}^{2\mathrm{i}\phi_c}$$
$$+\mu^2\left[\left(-\frac{39}{128} + \frac{3}{16}\ln\frac{8}{\mu}\right)\frac{1}{\rho_c} + \left(-\frac{1}{16}\ln\frac{8}{\mu\rho_c} + \frac{1}{8}S_n + \frac{5}{64}\right)\rho_c\right]\mathrm{e}^{-\mathrm{i}\phi_c}$$
$$+\mu^2\left[\left(-\frac{4n^2-1}{64} - \frac{79}{128}\right)\frac{1}{\rho_c} + \left(\frac{2n^2-1}{8}\ln\frac{8}{\mu\rho_c} - \frac{2n^2-1}{4}S_n + \frac{n^2+1}{8}\right)\rho_c\right]\mathrm{e}^{\mathrm{i}\phi_c}$$
$$+O(\mu^2)\,\mathrm{e}^{3\mathrm{i}\phi_c} + O(\mu^3), \quad m = 1, \qquad \text{(B 12}a\text{)}$$

$$\Phi = -\frac{1}{2m\rho_c^m}\mathrm{e}^{\mathrm{i}m\phi_c} - \frac{\mu}{16}\left(\frac{2}{m\rho_c^{m-1}} + \frac{3+m}{m+1}\frac{1}{\rho_c^{m+1}}\right)\mathrm{e}^{\mathrm{i}(m+1)\phi_c} - \frac{\mu}{16}\left(\frac{2}{m} + \frac{3-m}{m-1}\right)\frac{1}{\rho_c^{m-1}}\mathrm{e}^{\mathrm{i}(m-1)\phi_c}$$
$$-\mu^2\left[\left(\frac{56-m^2}{128m} + \frac{3+m}{64(m+1)} + \frac{4n^2-1}{32m(m+1)} + \frac{5}{32}\right)\frac{1}{\rho_c^m} + \left(\frac{3}{32m} + \frac{3-m}{64(m-1)} - \frac{4n^2-1}{32m(m-1)}\right)\right.$$
$$\left.\times\frac{1}{\rho_c^{m-2}}\right]\mathrm{e}^{\mathrm{i}m\phi_c} + O(\mu^2)\,\mathrm{e}^{\mathrm{i}(m-2)\phi_c} + O(\mu^2)\,\mathrm{e}^{\mathrm{i}(m+2)\phi_c} + O(\mu^3), \quad m \geqslant 2. \qquad \text{(B 12}b\text{)}$$

The multiplier $\exp(\mathrm{i}n\theta)$ is not written out further for brevity.

The velocity field $\boldsymbol{v} = \nabla\Phi$. Hence we get

$$V = 1 + \left(-\ln\frac{8}{\mu} + 2S_n + \frac{1}{4}\right)\frac{\mu}{2}\cos\psi + O(\mu^2), \quad m = 0, \qquad \text{(B 13}a\text{)}$$

$$V = \frac{1}{2}\mathrm{e}^{\mathrm{i}\psi} - \frac{\mu}{8}\mathrm{e}^{2\mathrm{i}\psi} + \mu^2\left(\frac{21}{64} - \frac{1}{4}\ln\frac{8}{\mu} + \frac{1}{8}S_n\right)\mathrm{e}^{-\mathrm{i}\psi}$$
$$+\frac{\mu^2}{8}\left[-\frac{n^2}{2} + 1 + (2n^2-1)\left(\ln\frac{8}{\mu} - 2S_n\right)\right]\mathrm{e}^{\mathrm{i}\psi} + O(\mu^2)\,\mathrm{e}^{3\mathrm{i}\psi} + O(\mu^3), \quad m = 1, \qquad (B\ 13b)$$

$$V = \frac{1}{2}\mathrm{e}^{\mathrm{i}m\psi} - \frac{\mu}{8m}(\mathrm{e}^{\mathrm{i}(m+1)\psi} + \mathrm{e}^{\mathrm{i}(m-1)\psi})$$
$$+\mu^2\frac{2n^2-m^2}{8m(m-1)(m+1)}\mathrm{e}^{\mathrm{i}m\psi} + O(\mu^2)\,\mathrm{e}^{\mathrm{i}(m\pm 2)\psi} + O(\mu^3), \quad m \geqslant 2, \qquad \text{(B 13}c\text{)}$$

where V is the σ-component of the field $\boldsymbol{v}$ at the boundary $\sigma = 1$.

Secondly, find the velocity field $\boldsymbol{v}$ generated by the axisymmetrical vorticity:

$$\Omega^\sigma = 0, \quad \Omega^\psi = 0, \quad \Omega^s = \delta(\sigma - \sigma_0)\,\mathrm{e}^{\mathrm{i}m\psi}, \qquad \text{(B 14)}$$

where $\sigma_0 \leqslant 1$, $\delta(x)$ is Dirac's delta-function. This task is similar to the previous one. The velocity field in the region $\sigma > 1$ is expressed as follows:

$$\boldsymbol{v} = \nabla \times \boldsymbol{A}, \quad \boldsymbol{A} = \frac{1}{4\pi}\int\frac{\boldsymbol{\Omega}(\boldsymbol{r}')\,\mathrm{d}\boldsymbol{r}'}{|\boldsymbol{r} - \boldsymbol{r}'|}. \qquad \text{(B 15)}$$

Substituting (B 3) into (B 15) we obtain for the covariant component:

$$A_s = \frac{R(1 - \mu_c\rho_c\cos\phi_c)^{1/2}}{2\pi R_c}$$
$$\times\int_0^1\int_0^{2\pi}\delta(\sigma' - \sigma_0)\,(1 - \mu_c\rho_c'\cos\phi_c')^{1/2}\,\mathrm{e}^{\mathrm{i}m\psi'}\,P(\sigma, \psi, \sigma', \psi')\,\sigma'\,\mathrm{d}\sigma'\,\mathrm{d}\psi', \qquad \text{(B 16)}$$

where

$$P=\int_0^{\pi}\frac{\cos\alpha\,\mathrm{d}\alpha}{[2(1-\cos\alpha)+\gamma^2]^{1/2}},$$

γ is determined by (B 4). Using (B 10) and integrating over σ' and ψ' we obtain the value of A_s. The velocity field $v^{\sigma}=(1/\sigma)\,\partial A_s/\partial\psi$. Hence we get at $\sigma=1$

$$V=\tfrac{1}{2}\mathrm{i}\sigma_0\,\mathrm{e}^{\mathrm{i}\psi}+\mathrm{i}\mu[\tfrac{5}{8}\sigma_0-1/\sigma_0]\,\sigma_0^2\,\mathrm{e}^{2\mathrm{i}\psi}+O(\mu^2),\quad m=1, \tag{B 17a}$$

$$V=\tfrac{1}{2}\mathrm{i}\sigma_0^m\,\mathrm{e}^{\mathrm{i}m\psi}+\mathrm{i}\mu\left[\frac{3m+2}{8}\sigma_0-\frac{(3m+1)\,(m+1)}{8m}\frac{1}{\sigma_0}\right]\sigma_0^{m+1}\,\mathrm{e}^{\mathrm{i}(m+1)\psi}$$
$$-\mathrm{i}\mu\frac{(2m-1)}{8m}\sigma_0^m\,\mathrm{e}^{\mathrm{i}(m-1)\psi}+O(\mu^2),\quad m\geqslant 2, \tag{B 17b}$$

where V is the σ-component of the field $\boldsymbol{v}$ at the boundary $\sigma=1$.

The results obtained in this Appendix are readily generalized for the case $m\leqslant -1$ by complex conjugation and the simultaneous replacement $m\to -m$.

Appendix C

The basic displacements are the particular solutions of (4.10). Let $n\geqslant 1$, and frequency $\omega=l/2+\omega'$, where $\omega'\leqslant O(\mu)$, $l\geqslant 2$. The system of basic displacements is chosen in the form

$$\epsilon^{\sigma}_{(l)}=\mathrm{i}\left[-\frac{l\omega'}{\mu n\sigma}(1-\omega')\,J_l(a)-\omega'\,J_{l+1}(a)+O(\mu\omega')\right]\mathrm{e}^{\mathrm{i}l\psi}$$
$$+\mathrm{i}\left[(l-1)\,(3l-2)\frac{\omega'^3}{\mu n^2\sigma}J_{l-1}(a)-(l-1)\frac{\omega'^2}{2n}J_l(a)+O(\mu\omega'^2)\right]\mathrm{e}^{\mathrm{i}(l-1)\psi}$$
$$+\mathrm{i}\left[-(l+1)\,(3l+2)\frac{\omega'^3}{\mu n^2\sigma}J_{l+1}(a)+(l+1)\frac{\omega'^2}{2n}J_l(a)+O(\mu\omega'^2)\right]\mathrm{e}^{\mathrm{i}(l+1)\psi}+\mathrm{O}(\mu\omega'^2), \tag{C 1a}$$

$$\epsilon^{\psi}_{(l)}=\left[\frac{l\omega'}{\mu n\sigma^2}(1-\omega')\,J_l(a)-\frac{1}{\sigma}\left(1+\frac{21}{16}\frac{l\omega'}{n^2}\right)J_{l+1}(a)+O(\mu^2)\right]\mathrm{e}^{\mathrm{i}l\psi}$$
$$+\left[-(l-1)\,(3l-2)\frac{\omega'^3}{\mu n^2\sigma^2}J_{l-1}(a)+\frac{\mu\omega'}{2}J_{l-1}(a)+\frac{(5l-3)\,\omega'^2}{2n\sigma}J_l(a)+O(\mu^2\omega')\right]$$
$$\times\,\mathrm{e}^{\mathrm{i}(l-1)\psi}$$
$$+\left[-(l+1)\,(3l+2)\frac{\omega'^3}{\mu n^2\sigma^2}J_{l+1}(a)+\frac{\mu\omega'}{2}J_{l+1}(a)+\frac{(5l+3)\omega'^2}{2n\sigma}J_l(a)+O(\mu^2\omega')\right]$$
$$\times\,\mathrm{e}^{\mathrm{i}(l+1)\psi}+O(\mu^2\omega'), \tag{C 1b}$$

$$\epsilon^{s}_{(l)}=J_l(a)\,[1+O(\mu^2)]\,\mathrm{e}^{\mathrm{i}l\psi}+[2\omega'\mu\sigma J_l(a)+O(\mu^2\omega')]\,\mathrm{e}^{\mathrm{i}(l-1)\psi}$$
$$-[2\omega'\mu\sigma J_l(a)+O(\mu^2\omega')]\,\mathrm{e}^{\mathrm{i}(l+1)\psi}+O(\mu^2\omega'), \tag{C 1c}$$

where

$$a=\frac{\mu n\sigma}{\omega'}\left[1+\frac{21}{16}\frac{l\omega'}{n^2}+\frac{163}{192}\mu^2\sigma^2+O(\mu\omega',\mu^3)\right].$$

The different harmonics in (C 1) are calculated with different accuracy. The values of rejected terms in the lth and $(l\pm 1)$th harmonics are presented in square brackets. The rest of the harmonics have values of no greater than $O(\mu\omega'^2)$ for the σ-component and less than $O(\mu^2\omega')$ for other components. Notation of this type will be also used below:

$$\epsilon^{\sigma}_{(l+1)} = \left[\sigma^l + \mu^2\left(\frac{(7l^2+21l-2)}{64(l+2)} - \frac{(3l+5)\,n^2}{4(l+2)\,(l+1)}\right)\sigma^{l+2} + O(\mu^3, \omega'^2)\right]\mathrm{e}^{\mathrm{i}(l+1)\psi} - \left[\mu\frac{2l+3}{4(l+1)}\sigma^{l+1} + O(\mu^2)\right]\mathrm{e}^{\mathrm{i}(l+2)\psi} - \left[\frac{l(3l+2)\,\omega'}{2n^2\mu}\sigma^{l-1} + \frac{(2l+1)}{2(l+1)}\mu\sigma^{l+1} + O(\mu^2, \omega')\right]\mathrm{e}^{\mathrm{i}l\psi} + O(\mu^2), \tag{C 2a}$$

$$\epsilon^{\psi}_{(l+1)} = \mathrm{i}\left[\sigma^{l-1} + \mu^2\left(\frac{(l+3)\,(7l^2+21l-2)}{64(l+1)\,(l+2)} - \frac{(3l+7)\,n^2}{4(l+2)\,(l+1)}\right)\sigma^{l+1} + O(\mu^3, \omega'^2)\right]\mathrm{e}^{\mathrm{i}(l+1)\psi} - \mathrm{i}\left[\mu\frac{2l+3}{4(l+1)}\sigma^{l} + O(\mu^2)\right]\mathrm{e}^{\mathrm{i}(l+2)\psi} + \mathrm{i}\left[-\frac{l(3l+2)\,\omega'}{2n^2\mu}\sigma^{l-2} + \frac{l}{2(l+1)}\mu\sigma^{l} + O(\mu^2, \omega')\right]\mathrm{e}^{\mathrm{i}l\psi} + O(\mu^2), \tag{C 2b}$$

$$\epsilon^{s}_{(l+1)} = -\mathrm{i}\left[\frac{\mu n}{l+1}\sigma^{l+1} + O(\mu^2)\right]\mathrm{e}^{\mathrm{i}(l+1)\psi} - \mathrm{i}\left[\frac{(3l+2)}{2n}\sigma^{l} + O(\mu)\right]\mathrm{e}^{\mathrm{i}l\psi} + O(\mu^2); \tag{C 2c}$$

$$\left.\begin{aligned} &\epsilon^{\sigma}_{(m)} = \sigma^{m-1}\,\mathrm{e}^{\mathrm{i}m\psi} + O(\mu), \\ &\epsilon^{\psi}_{(m)} = \mathrm{i}\sigma^{m-2}\,\mathrm{e}^{\mathrm{i}m\psi} + O(\mu), \quad \epsilon^{s}_{(m)} = O(\mu), \quad m \geqslant 1, \quad m \neq l, l+1; \end{aligned}\right\} \tag{C 3}$$

$$\epsilon^{\sigma}_{(m)} = \sigma^{|m|-1}\,\mathrm{e}^{\mathrm{i}m\psi} + O(\mu), \quad \epsilon^{\psi}_{(m)} = -\mathrm{i}\sigma^{|m|-2}\,\mathrm{e}^{\mathrm{i}m\psi} + O(\mu), \quad \epsilon^{s}_{(m)} = O(\mu), \quad m \leqslant -1; \tag{C 4}$$

$$\epsilon^{\sigma}_{(m)} = \sigma\,\mathrm{e}^{\mathrm{i}0\psi} + O(\mu), \quad \epsilon^{\psi}_{(m)} = -\mathrm{i}\frac{2}{l}\mathrm{e}^{\mathrm{i}0\psi} + O(\mu), \quad \epsilon^{s}_{(m)} = \mathrm{i}\frac{2}{\mu n}\mathrm{e}^{\mathrm{i}0\psi} + O(1), \quad m = 0. \tag{C 5}$$

The basic displacements at $l = 1$ are described by (C 1)–(C 4), the only difference being a different expression for $\boldsymbol{\varepsilon}_{(0)}$ and more precise evaluations of the zero harmonic of the σ-components. That is

$$\epsilon^{\sigma}_{(0)} = [\mu + O(\mu^2)]\,\mathrm{e}^{\mathrm{i}\psi} - [\mu^2 n^2\sigma + O(\mu^3)]\,\mathrm{e}^{\mathrm{i}0\psi} + O(\mu^3), \tag{C 6a}$$

$$\epsilon^{\psi}_{(0)} = \mathrm{i}\left[\mu\frac{1}{\sigma} + O(\mu^2)\right]\mathrm{e}^{\mathrm{i}\psi} + \mathrm{i}[2n^2\mu^2 + O(\mu^3)]\,\mathrm{e}^{\mathrm{i}0\psi} + O(\mu^3), \tag{C 6b}$$

$$\epsilon^{s}_{(0)} = -\mathrm{i}2n\mu\,\mathrm{e}^{\mathrm{i}0\psi} + O(\mu^2); \tag{C 6c}$$

$$\epsilon^{\sigma}_{(1)} = \mathrm{i}\left[-\frac{\omega'}{\mu n\sigma}(1-\omega')\,J_1(a) - \omega'\,J_2(a) + O(\mu\omega')\right]\mathrm{e}^{\mathrm{i}\psi} + \mathrm{i}[-2\mu\omega'^2\sigma J_2(a) + O(\mu^2\omega'^2)]\,\mathrm{e}^{\mathrm{i}0\psi} + \mathrm{i}\left[-\frac{10\omega'^3}{\mu n^2\sigma}J_2(a) + \frac{\omega'^2}{n}J_1(a) + O(\mu\omega'^2)\right]\mathrm{e}^{\mathrm{i}2\psi} + O(\mu\omega'^2), \tag{C 7}$$

$$\epsilon^{\sigma}_{(2)} = \left[\sigma + \mu^2\left(\frac{13}{96} - \frac{n^2}{3}\right)\sigma^3 + O(\mu^3, \omega'^2)\right]\mathrm{e}^{\mathrm{i}2\psi} - \left[\mu\frac{5}{8}\sigma^2 + O(\mu^2)\right]\mathrm{e}^{\mathrm{i}3\psi} - \left[\frac{5\omega'}{2n^2\mu} + \frac{3}{4}\mu\sigma^2 + O(\mu^2, \omega')\right]\mathrm{e}^{\mathrm{i}\psi} + O(\mu^3)\,\mathrm{e}^{\mathrm{i}0\psi} + O(\mu^2), \tag{C 8}$$

$$\epsilon^{\sigma}_{(m)} = \sigma^{|m|-1}\,\mathrm{e}^{\mathrm{i}m\psi} + O(\mu^2)\,\mathrm{e}^{\mathrm{i}0\psi} + O(\mu), \quad m \neq 0, 1, 2. \tag{C 9}$$

The basic displacements at $l=0$ are described by (C 1)–(C 4). However, we shall use more precise evaluations for displacements $\boldsymbol{\varepsilon}_{(\pm 1)}$. These displacement fields are

$$\epsilon^{\sigma}_{(\pm 1)} = \left[1+\mu^2\left(-\frac{1}{64}-\frac{5}{8}n^2\right)\sigma^2+O(\mu^3)\right]\mathrm{e}^{\pm \mathrm{i}\psi}-\left[\mu\frac{3}{4}\sigma+O(\mu^2)\right]\mathrm{e}^{\pm 2\mathrm{i}\psi} -\left[\frac{1}{2}\mu\sigma+O(\mu^2)\right]\mathrm{e}^{0\mathrm{i}\psi}+\left[\mu^2\frac{3}{32}\sigma^2+O(\mu^3)\right]\mathrm{e}^{\mp \mathrm{i}\psi}+O(\mu^2), \quad \text{(C 10}a\text{)}$$

$$\epsilon^{\psi}_{(\pm 1)} = \pm\mathrm{i}\left[\frac{1}{\sigma}+\mu^2\left(-\frac{3}{64}-\frac{7}{8}n^2\right)\sigma+O(\mu^3)\right]\mathrm{e}^{\pm \mathrm{i}\psi}\mp\mathrm{i}\left[\mu\frac{3}{4}+O(\mu^2)\right]\mathrm{e}^{\pm 2\mathrm{i}\psi} +O(\mu^2)\,\mathrm{e}^{0\mathrm{i}\psi}\mp\mathrm{i}\left[\frac{9}{32}\mu^2\sigma+O(\mu^3)\right]\mathrm{e}^{\mp \mathrm{i}\psi}+O(\mu^2), \quad \text{(C 10}b\text{)}$$

$$\epsilon^{s}_{(\pm 1)} = -\mathrm{i}[\mu n\sigma+O(\mu^2)]\,\mathrm{e}^{\pm \mathrm{i}\psi}-\mathrm{i}\left[\frac{1}{n}+O(\mu)\right]\mathrm{e}^{0\mathrm{i}\psi}+O(\mu^2). \quad \text{(C 10}c\text{)}$$

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